CHAPTER 1

BASIC IDEAS OF SPECIAL FUNCTIONS AND STATISTICAL DISTRIBUTIONS

[*This chapter is based on the lectures of Professor A.M. Mathai of McGill University, Canada (Director of the 4th SERC School).*]

1.0. Introduction

Some preliminaries of special functions and statistical distributions are given here. Details are available from the following sources, which are accessible to the participants of the 4^{th} SERC School:

(1) *Notes of the* 2^{nd} *SERC School*. (Publication No 31 of the Centre for Mathematical Sciences (CMS)), 2000.
(2) *Notes of the* 3^{rd} *SERC School*. (Publication No 32 of the Centre for Mathematical Sciences (CMS)), 2005.
(3) Mathai, A.M. (1993). *"A Handbook of Generalized Special Functions for Statistical and Physical Sciences"*, Oxford University Press, Oxford, U.K.
(4) Mathai, A.M. and Saxena, R.K. (1978). *"The H-Function with Applications in Statistics and Other Disciplines"*, Wiley Halsted, New York.
(5) Mathai, A.M. and Saxena (1973). *"Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences"*, Lecture Notes No 348, Springer-Verlag, Heidelberg, Germany.

Notation 1.0.1. **Pochammer Symbol**

$$(b)_r = b(b+1)\cdots(b+r-1),\ (b)_0 = 1,\ b \neq 0. \tag{1.0.1}$$

For example,

$$\left(-\frac{1}{4}\right)_2 = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}+1\right) = -\frac{3}{16}; \quad (-2)_3 = (-2)(-1)(0) = 0;$$
$$\left(\frac{1}{3}\right)_4 = \left(\frac{1}{3}\right)\left(\frac{1}{3}+1\right)\left(\frac{1}{3}+2\right)\left(\frac{1}{3}+3\right) = \frac{280}{81}; (7)_0 = 1; (0)_5 = \text{not defined.}$$

The following general property holds for m, n non-negative integers

$$(a)_{m+n} = (a)_m(a+m)_n = (a)_n(a+n)_m. \tag{1.0.2}$$

Notation 1.0.2. **Factorial n or n factorial**

$$n! = (1)(2)\cdots(n),\ 0! = 1 \text{ (convention).} \tag{1.0.3}$$

For example,

$$\frac{1}{3}! = \text{not defined}; (-5)! = \text{not defined}; 1! = 1; 0! = 1 \quad \text{(convention)};$$
$$3! = (1)(2)(3) = 6; \quad 4! = (1)(2)(3)(4) = 24.$$

Notation 1.0.3. **Number of Combinations of n taken r at a time**

$$\binom{n}{r} = \text{number of subsets of } r \text{ distinct objects from a set of } n \text{ distinct objects}$$
$$= \frac{n(n-1)\cdots(n-(r-1))}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}, \quad 0 \le r \le n. \tag{1.0.4}$$

For example,

$$\binom{4}{1} = \frac{4}{1!} = 4;\ \binom{4}{0} = \frac{4!}{0!(4-0)!} = \frac{4!}{1\ (4!)} = 1;\ \binom{4}{4} = \frac{(4)(3)(2)(1)}{4!} = 1;$$

$$\binom{n}{1} = \frac{n}{1!} = \binom{n}{n-1} = \frac{n(n-1)\cdots(n-(n-1))}{(n-1)!} = n;\binom{n}{r} = \binom{n}{n-r},\quad r = 0, 1, \cdots, n;$$

$$\binom{n}{0} = \binom{n}{n} = 1;\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1};\binom{1/4}{1} = \text{not defined as a combination };$$

$$\binom{-3}{2} = \text{not defined as a combination };\binom{0}{2} = \text{not defined as a combination };$$

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{(-1)^r(-n)(-n+1)\cdots(-n+r-1)}{r!}$$

$$= \frac{(-1)^r(-n)_r}{r!}. \tag{1.0.5}$$

If $\binom{n}{r}$ is interpreted not as the number of combinations but as in equation (1.0.5) then one can give interpretations when n is not a positive integer. For example

$$\binom{-1/3}{2} = \frac{(-1)^2(\frac{1}{3})_2}{2!} = \frac{(-1)^2}{2!}\Big(\frac{1}{3}\Big)\Big(\frac{1}{3}+1\Big) = \frac{1}{2!}\frac{4}{9} = \frac{2}{9};$$

$$\binom{1/2}{2} = \frac{(-1)^2(-\frac{1}{2})_2}{2!} = \frac{(-1)^2}{2!}\Big(-\frac{1}{2}\Big)\Big(-\frac{1}{2}+1\Big) = \frac{1}{2!}\frac{(-1)}{4} = -\frac{1}{8}.$$

1.1. Gamma Function

Notation 1.1.1. $\Gamma(z) =$ **gamma** z

A gamma function $\Gamma(z)$ can be defined in many ways. $\Gamma(z)$ exists for all values of z, negative, positive and complex values of z, except at $z = 0, -1, -2, \cdots$. Also $\Gamma(z)$ has an integral representation for $\Re(z) > 0$ where $\Re(\cdot)$ means the real part of $(\cdot)$. Thus we may note for example that $\Gamma(5)$ exists ; $\Gamma(-\frac{1}{3})$ exists ; $\Gamma(\frac{2}{5})$ exists; $\Gamma(0)$ does not exist; $\Gamma(-3)$ does not exist; $\Gamma(-\frac{7}{2})$ exists. Some of the definitions of $\Gamma(z)$ are the following:

Definition 1.1.1.

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\ n^z}{z(z+1)\cdots(z+n)},\quad z \neq 0, -1, -2, \cdots \tag{1.1.1}$$

Definition 1.1.2.

$$\Gamma(z) \;=\; z^{-1}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}. \tag{1.1.2}$$

Definition 1.1.3.

$$\frac{1}{\Gamma(z)} \;=\; z\lim_{n\to\infty}\left\{n^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)\right\}. \tag{1.1.3}$$

Definition 1.1.4.

$$\frac{1}{\Gamma(z)} \;=\; z\mathrm{e}^{\gamma z}\prod_{n=1}^{\infty}\left[(1+\frac{z}{n})\mathrm{e}^{-\frac{z}{n}}\right] \tag{1.1.4}$$

where γ is the Euler's constant, defined as follows:

Notation 1.1.2. Euler's constant γ

$$\gamma = \lim_{n\to\infty}\left\{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right\} \approx 0.5772156649015328606065 12. \tag{1.1.5}$$

Definition 1.1.5.

$$\Gamma(z) \;=\; p^{z}\int_{0}^{\infty} t^{z-1}\mathrm{e}^{-\mathrm{pt}}\mathrm{dt},\, \Re(\mathrm{p}) > 0, \Re(\mathrm{z}) > 0. \tag{1.1.6}$$

Definition 1.1.6.

$$\frac{1}{\Gamma(z)} \;=\; \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-z}\mathrm{e}^{t}\mathrm{d}t,\; c > 0, \Re(\mathrm{z}) > 0,\; i = \sqrt{-1} \tag{1.1.7}$$

where π is the mathematical constant,

$$\pi \approx 3.14159265358979323846264 3. \tag{1.1.8}$$

1.1.1. Some basic properties of gamma functions

From all the definitions of $\Gamma(z)$ it is not difficult to show that

$$\Gamma(z) = (z-1)\Gamma(z-1) \tag{1.1.9}$$

whenever the gammas are defined. It is obvious from the integral representation in (1.1.6). Take $p = 1$ and integrate by parts by using the formula

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

and by taking $\mathrm{d}v = \mathrm{e}^{-t}$ and $u = t^{z-1}$. If the integral representation is used then we need the conditions $\Re(z) > 0, \Re(z-1) > 0$ which means $\Re(z) > 1$. This restriction is not needed for the Definitions 1.1.1 - 1.1.4. [verification and derivation of the result in (1.1.9) by using the Definitions 1.1.1 - 1.1.4 are left to the reader]. Continuing the process in (1.1.9) we have the following :

$$\Gamma(z) = (z-1)(z-2)\cdots(z-r)\Gamma(z-r) \tag{1.1.10}$$

whenever the gammas exist. As a consequence of (1.1.10) we may note that for $n = 1, 2, \cdots$

$$\Gamma(n) = (n-1)(n-2)\cdots 1\Gamma(1) = (n-1)(n-2)\cdots 1 = (n-1)! \tag{1.1.11}$$

since $\Gamma(1) = 1$. Thus, $\Gamma(z)$ can be looked upon as a generalization of $(z-1)!$ Thus for example,

$$\Gamma(5) = 4! = 24; \Gamma(-2) = \text{not defined};\quad \Gamma\left(\frac{5}{2}\right) = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\Gamma\left(\frac{1}{2}\right);$$
$$\Gamma\left(\frac{1}{2}\right) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\Gamma\left(-\frac{7}{2}\right) = \frac{105}{32}\Gamma\left(-\frac{7}{2}\right)$$
$$\Rightarrow \Gamma\left(-\frac{7}{2}\right) = \frac{32}{105}\Gamma\left(\frac{1}{2}\right).$$

Thus whenever $\Gamma(z)$ is defined we can write it in the form

$$\Gamma(z) = (\text{a few factors})\ \Gamma(\alpha),\ \ 0 < \alpha \leq 1. \tag{1.1.12}$$

But $\Gamma(\alpha)$ for $0 < \alpha \leq 1$ is extensively tabulated. Hence for computational purposes we may use the formula in (1.1.10) and the extensive tables for $\Gamma(\alpha)$ for $0 < \alpha \leq 1$.

Example 1.1.1. Evaluate $\Gamma\left(\frac{51}{2}\right)\ \Gamma\left(-\frac{27}{2}\right)$.

Solution 1.1.1. By using (1.1.10) we have the following:

$$\Gamma\Big(\frac{1}{2}\Big) = \Big(-\frac{1}{2}\Big)\Big(-\frac{3}{2}\Big)\cdots\Big(-\frac{27}{2}\Big)\,\Gamma\Big(-\frac{27}{2}\Big) = \frac{(-1)^{14}(1)(3)\cdots(27)}{2^{14}}\,\Gamma\Big(-\frac{27}{2}\Big) \Rightarrow$$

$$\Gamma\Big(-\frac{27}{2}\Big) = \frac{2^{14}}{(1)(3)\cdots(27)}\,\Gamma\Big(\frac{1}{2}\Big)$$

$$\Gamma\Big(\frac{51}{2}\Big) = \Big(\frac{49}{2}\Big)\Big(\frac{47}{2}\Big)\cdots\Big(\frac{1}{2}\Big)\,\Gamma\Big(\frac{1}{2}\Big).$$

Hence

$$\Gamma\Big(\frac{51}{2}\Big)\,\Gamma\Big(-\frac{27}{2}\Big) = \frac{(49)(47)\cdots(27)(25)\cdots(1)\,\Gamma(\frac{1}{2})}{2^{26}}\,\frac{2^{14}}{(1)(3)\cdots(27)}\,\Gamma\Big(\frac{1}{2}\Big)$$

$$= \frac{(49)(47)\cdots(28)}{2^{12}}\,\Big[\Gamma(\frac{1}{2})\Big]^2.$$

Direct computation of this quantity will overflow in the computer. Hence take logarithms, simplify and then take antilogarithm to obtain the exact result. It can be shown that $\Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}$ where π is the mathematical constant and hence one may use (1.1.8) while computing $\Gamma\Big(\frac{1}{2}\Big)$.

Example 1.1.2. Show that $\Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}$.

Solution 1.1.2. A simple proof can be given with the help of the integral representation for gamma functions.

$$\Big[\Gamma\Big(\frac{1}{2}\Big)\Big]^2 = \Gamma\Big(\frac{1}{2}\Big)\Gamma\Big(\frac{1}{2}\Big) = \Big[\int_0^\infty x^{\frac{1}{2}-1}e^{-x}dx\Big]\Big[\int_0^\infty y^{\frac{1}{2}-1}e^{-y}dy\Big]$$

$$= \int_0^\infty\int_0^\infty x^{-\frac{1}{2}}y^{-\frac{1}{2}}e^{-(x+y)}dxdy.$$

Put $x = r\cos^2\theta$, $y = r\sin^2\theta$, $0 \le r < \infty$, $0 \le \theta \le \frac{\pi}{2}$. Then the Jacobian is $2r\sin\theta\cos\theta$. Then,

$$\begin{aligned}\left[\Gamma\left(\frac{1}{2}\right)\right]^2 &= \int_{r=0}^{\infty}\int_{\theta=0}^{\frac{\pi}{2}}(r\cos^2\theta)^{-\frac{1}{2}}(r\sin^2\theta)^{-\frac{1}{2}}2r\cos\theta\sin\theta e^{-r}\mathrm{d}r\wedge\mathrm{d}\theta \\ &= \left(2\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\right)\left(\int_0^{\infty}e^{-r}\mathrm{d}r\right) = \pi \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad (1.1.13)\end{aligned}$$

where $\mathrm{d}r \wedge \mathrm{d}\theta$ denotes the wedge product or skew symmetric product of the differentials $\mathrm{d}r$ and $\mathrm{d}\theta$.

Also one can give a representation of the Pochammer symbol in terms of gamma functions .

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \qquad (1.1.14)$$

whenever the gammas exist.

1.1.2. Wedge product and Jacobians of transformations

Notation 1.1.3. $\wedge =$ wedge product, $\mathrm{d}x \wedge \mathrm{d}y =$ wedge product of $\mathrm{d}x$ and $\mathrm{d}y$.

Definition 1.1.7.

$$\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x \Rightarrow \mathrm{d}x \wedge \mathrm{d}x = -\mathrm{d}x \wedge \mathrm{d}x = 0.$$

Thus, a wedge product is a skew symmetric product. As a consequence of Definition 1.1.7 we can evaluate the Jacobians when transforming a set of variables to another set of variables. As an example, let us consider two scalar functions of two real scalar variables x_1 and x_2. Let

$$y_1 = f_1(x_1, x_2) \text{ and } y_2 = f_2(x_1, x_2).$$

Then

$$\mathrm{d}y_1 = \frac{\partial f_1}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_1}{\partial x_2}\mathrm{d}x_2 \text{ and } \mathrm{d}y_2 = \frac{\partial f_2}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_2}{\partial x_2}\mathrm{d}x_2$$

where $\frac{\partial}{\partial x}$ denotes the partial derivative operator. Then

$$
\begin{aligned}
\mathrm{d}y_1 \wedge \mathrm{d}y_2 &= \left[\frac{\partial f_1}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_1}{\partial x_2}\mathrm{d}x_2\right] \wedge \left[\frac{\partial f_2}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_2}{\partial x_2}\mathrm{d}x_2\right] \\
&= \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_1}\mathrm{d}x_1 \wedge \mathrm{d}x_1 + \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}\mathrm{d}x_1 \wedge \mathrm{d}x_2 \\
&\quad + \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}\mathrm{d}x_2 \wedge \mathrm{d}x_1 + \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_2}\mathrm{d}x_2 \wedge \mathrm{d}x_2 \\
&= 0 + \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}\mathrm{d}x_2 \wedge \mathrm{d}x_1 + 0 \\
&\quad \text{since } \ \mathrm{d}x_1 \wedge \mathrm{d}x_1 = 0 \text{ and } \mathrm{d}x_2 \wedge \mathrm{d}x_2 = 0 \\
&= \left[\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}\right]\mathrm{d}x_1 \wedge \mathrm{d}x_2 = J\mathrm{d}x_1 \wedge \mathrm{d}x_2 \\
&\quad \text{since } \ \mathrm{d}x_2 \wedge \mathrm{d}x_1 = -\mathrm{d}x_1 \wedge \mathrm{d}x_2
\end{aligned}
$$

where J is the Jacobian given by the expression

$$
J = \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix}. \tag{1.1.15}
$$

Observe that a 2×2 determinant is evaluated as

$$
\begin{vmatrix} a & b \\ c & d \end{vmatrix} = (a)(d) - (c)(b).
$$

where $\|[.]\|$ denotes the determinant of the matrix [.].

Example 1.1.3. Evaluate the Jacobian in the transformation $y_1 = x_1 + x_2$ and $y_2 = x_1$.

Solution 1.1.3. The Jacobian J is given by

$$
J = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = (1)(0) - (1)(1) = -1
$$

where

$$\frac{\partial f_1}{\partial x_1} = \frac{\partial}{\partial x_1}(x_1 + x_2) = 1, \frac{\partial}{\partial x_2}(x_1 + x_2) = 1, \frac{\partial}{\partial x_1}(x_1) = 1, \frac{\partial}{\partial x_2}(x_1) = 0.$$

Example 1.1.4. Evaluate the Jacobian in the transformation $x = r\cos^2\theta, y = r\sin^2\theta$.

Solution 1.1.4. The partial derivatives with respect to r and θ are the following:

$$\frac{\partial x}{\partial r} = \cos^2\theta, \frac{\partial x}{\partial \theta} = -2r\cos\theta\sin\theta, \quad \frac{\partial y}{\partial r} = \sin^2\theta, \quad \frac{\partial y}{\partial \theta} = 2r\sin\theta\cos\theta.$$

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos^2\theta & -2r\cos\theta\sin\theta \\ \sin^2\theta & 2r\cos\theta\sin\theta \end{vmatrix} \\ &= \cos^2\theta[2r\cos\theta\sin\theta] + \sin^2\theta[2r\cos\theta\sin\theta] \\ &= 2r\cos\theta\sin\theta[\cos^2\theta + \sin^2\theta] = 2r\cos\theta\sin\theta. \end{aligned}$$

This means,

$$\mathrm{d}x \wedge \mathrm{d}y = J\mathrm{d}r \wedge \mathrm{d}\theta = 2r\cos\theta\sin\theta \;\; \mathrm{d}r \wedge \mathrm{d}\theta. \tag{1.1.16}$$

Note that the various steps in the solution of Example 1.1.2 are done with the help of (1.1.16).

1.1.3. Multiplication formula for gamma functions

$$\Gamma(mz) = (2\pi)^{\frac{1-m}{2}} m^{mz-\frac{1}{2}} \Gamma(z)\Gamma\Big(z + \frac{1}{m}\Big) \cdots \Gamma\Big(z + \frac{m-1}{m}\Big), \; m = 1, 2, \cdots \tag{1.1.17}$$

For $m = 2$ we obtain the duplication formula for gamma functions, namely,

$$\Gamma(2z) = (2\pi)^{\frac{1-2}{2}} 2^{2z-\frac{1}{2}} \Gamma(z)\Gamma\Big(z + \frac{1}{2}\Big) = \pi^{-\frac{1}{2}} 2^{2z-1} \Gamma(z)\Gamma\Big(z + \frac{1}{2}\Big). \tag{1.1.18}$$

We may simplify gamma products with the help of (1.1.17). For example,

$$1 = \Gamma(1) = \Gamma\Big[2\Big(\frac{1}{2}\Big)\Big] = \pi^{-\frac{1}{2}} 2^{1-1} \Gamma\Big(\frac{1}{2}\Big)\Gamma\Big(\frac{1}{2} + \frac{1}{2}\Big) = \pi^{-\frac{1}{2}} \Gamma\Big(\frac{1}{2}\Big) \Rightarrow \Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}.$$

$$1 = \Gamma(1) = \Gamma\Big[3\Big(\frac{1}{3}\Big)\Big] = \pi^{\frac{1-3}{2}} 3^{1-\frac{1}{2}} \Gamma\Big(\frac{1}{3}\Big)\Gamma\Big(\frac{2}{3}\Big)\Gamma(1) \Rightarrow \Gamma\Big(\frac{1}{3}\Big)\Gamma\Big(\frac{2}{3}\Big) = \frac{2\pi}{\sqrt{3}}.$$

By using the product formulae for trigonometric functions we can establish the following results:

$$\Gamma(z)\Gamma(1-z) = \pi\ \pi z \tag{1.1.19}$$

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z}\pi z \tag{1.1.20}$$

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right) = \pi\ \sec \pi z. \tag{1.1.21}$$

1.1.4. Asymptotic formula for a gamma function

For $|z| \to \infty$ and α a bounded quantity, it can be shown that

$$\begin{aligned}\ln\Gamma(z+\alpha) =& \frac{1}{2}\ln(2\pi) + (z+\alpha-\frac{1}{2})\ln z - z\\ &+ \sum_{k=1}^{\infty}\frac{(-1)^{k+1}B_{k+1}(\alpha)}{k(k+1)z^k},\quad |\arg\,(z+\alpha)| \le \pi-\epsilon,\ \epsilon>0\end{aligned} \tag{1.1.22}$$

where $B_{k+1}(\alpha)$ is a Bernoulli polynomial. For the definitions and the list of the first few generalized Bernoulli polynomials, Bernoulli polynomials and Bernoulli numbers see Mathai (1993). The first part of (1.1.22) is known as *Stirling's approximation* for a gamma function, namely,

$$\Gamma(z+\alpha) \approx (2\pi)^{\frac{1}{2}} z^{z+\alpha-\frac{1}{2}} \mathrm{e}^{-z} \tag{1.1.23}$$

for $|z| \to \infty$ and α a bounded quantity.
For example, taking $z = 90$ and $\alpha = 0.5$ we have

$$\Gamma(90.5) \approx \sqrt{2\pi}(90)^{90}\ \mathrm{e}^{-90}.$$

For α and β bounded and $|z| \to \infty$ we have

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\alpha+\beta)} \approx \frac{(2\pi)^{\frac{1}{2}} z^{z+\alpha-\frac{1}{2}}\mathrm{e}^{-z}}{(2\pi)^{\frac{1}{2}} z^{z+\alpha+\beta}\mathrm{e}^{-z}} = z^{-\beta}\ . \tag{1.1.24}$$

Example 1.1.5. Evaluate the following integrals:

$$(1)\ \int_0^{\infty} x^4\ \mathrm{e}^{-x^8}\mathrm{d}x;\quad (2)\ \int_{-\infty}^{\infty} \mathrm{e}^{-2|x|}\mathrm{d}x;\quad (3)\ \int_0^{\infty} x^3\ \mathrm{e}^{-2x^{\frac{1}{2}}}\mathrm{d}x.$$

Solutions 1.1.1.

(1): Put $u = x^8 \Rightarrow x = u^{\frac{1}{8}} \Rightarrow \mathrm{d}x = \frac{1}{8}u^{\frac{1}{8}-1}\mathrm{d}u, \quad x^4 = u^{\frac{1}{2}}$.

$$\int_0^\infty x^4 e^{-x^8}\mathrm{d}x = \frac{1}{8}\int_0^\infty u^{\frac{1}{2}+\frac{1}{8}-1}\, e^{-u}\mathrm{d}u = \frac{1}{8}\Gamma\left(\frac{5}{8}\right).$$

(2): Since the integrand is an even function, $f(x) = f(-x)$, we have

$$\int_{-\infty}^\infty e^{-2|x|}\mathrm{d}x = 2\int_0^\infty e^{-2x}\mathrm{d}x = \int_0^\infty e^{-y}\mathrm{d}y, (2x = y) = 1.$$

(3): Put $y = 2x^{\frac{1}{2}} \Rightarrow x = \frac{y^2}{4} \Rightarrow \mathrm{d}x = \frac{2y}{4}\mathrm{d}y = \frac{y}{2}\mathrm{d}y$.

$$\int_0^\infty x^3 e^{-2x^{\frac{1}{2}}}\mathrm{d}x = \frac{1}{2(4^3)}\int_0^\infty y^7 e^{-y}\mathrm{d}y = \frac{1}{2(4^3)}\Gamma(8) = \frac{7!}{2(4^3)}$$
$$= \frac{315}{8}$$

Exercises 1.1.

1.1.1. Evaluate the following whenever they exist:

(a) $\left(-\frac{2}{3}\right)_4$; (b) $(-2)_3$; (c) $(1)_n$; (d) $(0)_3$.

1.1.2. Evaluate the following, interpreting as the number of combinations, whenever they exist:

(a) $\binom{1/3}{2}$; (b) $\binom{-1}{2}$; (c) $\binom{3}{5}$; (d) $\binom{5}{2}$; (e) $\binom{90}{4}$.

1.1.3. Evaluate the following in terms of $\Gamma(\alpha), 0 < \alpha \leq 1$.

(a) $\Gamma(-\frac{7}{2})$; (b) $\Gamma(-\frac{5}{4})$; (c) $\Gamma(\frac{5}{2})$; (d) $\Gamma(7)$.

1.1.4. Evaluate the following:

(a) $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})$; (b) $\Gamma(\frac{1}{6})\Gamma(\frac{5}{6})$.

1.1.5. Prove that Definitions 1.1.3 and Definition 1.1.4 are one and the same.

1.1.6. Prove that $z\Gamma(z) = \Gamma(z+1)$ by using Definitions 1.1.1 and 1.1.2.

1.1.7. Evaluate the following integrals:

(a) $\int_0^\infty x^{\frac{1}{2}} e^{-3x^5} dx$; (b) $\int_0^\infty x^{\alpha-1} e^{-ax^\delta} dx$ (state the conditions).

1.1.8. Evaluate the following integrals:

(a) $\int_{-\infty}^\infty e^{-3|x|} dx$; (b) $\int_{-\infty}^\infty |x|^{\alpha-1} e^{-a|x|^\delta} dx$ (state the conditions).

1.1.9. Show that for $\Re(\alpha) > 0$, $\Re(\beta) > 0$,

$$\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha+\beta)\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \Gamma(\alpha+\beta)\int_0^1 x^{\beta-1}(1-x)^{\alpha-1}dx.$$

1.1.10. Show that for $\Re(\alpha) > 0$, $\Re(\beta) > 0$,

$$\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha+\beta)\int_0^\infty x^{\alpha-1}(1+x)^{-(\alpha+\beta)}dx = \Gamma(\alpha+\beta)\int_0^\infty x^{\beta-1}(1+x)^{-(\alpha+\beta)}dx.$$

1.2. The Psi and Zeta Functions

The logarithmic derivative of a gamma function is the psi function and successive derivatives give generalized zeta functions.

Notation 1.2.1. $\psi(z)$: **psi z**

Definition 1.2.1.

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{1}{\Gamma(z)}\frac{d}{dz}\Gamma(z) \qquad (1.2.1)$$

$$\ln\Gamma(z) = \int_1^z \psi(x)dx.$$

By taking logarithm and then differentiating one can obtain many properties for psi functions from the corresponding properties of gamma functions. For example, from (1.1.10.) we have

$$\psi(z) = \frac{1}{z-1} + \frac{1}{z-2} + \cdots + \frac{1}{z-r} + \psi(z-r). \tag{1.2.2}$$

The following are some further properties :

$$\psi(z) = -\gamma - \frac{1}{z} + z\sum_{k=1}^{\infty}\frac{1}{k(z+k)} \tag{1.2.3}$$

$$\psi(z) = -\gamma + (z-1)\sum_{k=0}^{\infty}\frac{1}{(k+1)(z+k)} \tag{1.2.4}$$

$$\psi(1) = -\gamma \tag{1.2.5}$$

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2 \tag{1.2.6}$$

$$\psi(z) - \psi(1-z) = -\pi\cot\pi z \tag{1.2.7}$$

$$\psi\left(\frac{1}{2}+z\right) - \psi\left(\frac{1}{2}-z\right) = \pi\tan\pi z \tag{1.2.8}$$

where γ is the Euler's constant.

1.2.1. Generalized zeta function

Notation 1.2.2.

$\zeta(\rho, a)$: generalized zeta function

$\zeta(\rho)$: Riemann zeta function

Definition 1.2.2.

$$\zeta(\rho, a) = \sum_{k=0}^{\infty}\frac{1}{(k+a)^{\rho}},\ \Re(\rho) > 1,\ a \neq 0, -1, -2, \cdots \tag{1.2.9}$$

$$\zeta(\rho) = \sum_{k=1}^{\infty}\frac{1}{k^{\rho}},\ \Re(\rho) > 1\ . \tag{1.2.10}$$

For $\rho \leq 1$ the series is divergent. Successive derivatives of (1.2.4) yield the following results:

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\Gamma(z) = \frac{\mathrm{d}}{\mathrm{d}z}\psi(z) = \sum_{k=0}^{\infty}\frac{1}{(z+k)^2} = \zeta(2,z) \tag{1.2.11}$$

$$\frac{\mathrm{d}^r}{\mathrm{d}z^r}\ln\Gamma(z) = \frac{\mathrm{d}^{r-1}}{\mathrm{d}z^{r-1}}\psi(z) = \begin{cases}\psi(z), & \text{for } r = 1\\ (-1)^r(r-1)!\zeta(r,z), & \text{for } r \geq 2\end{cases}$$
$$= (-1)^r(r-1)!\sum_{k=0}^{\infty}\frac{1}{(z+k)^r}. \tag{1.2.12}$$

Explicit evaluations can be done in a few cases.

$$\zeta(2) = \zeta(2,1) = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6} \tag{1.2.13}$$

$$\zeta(4) = \zeta(4,1) = \sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90} \tag{1.2.14}$$

$$\zeta(2n) = \zeta(2n,1) = \sum_{k=1}^{\infty}\frac{1}{k^{2r}} = \frac{(-1)^{r+1}(2\pi)^{2r}}{2(2r)!}B_{2r} \tag{1.2.15}$$

where B_{2r} is a Bernoulli number. For these and other results see Mathai (1993).

Exercises 1.2.

1.2.1. Prove formula (1.2.4) by using (1.1.10).

1.2.2. Prove formula (1.2.3).

1.2.3. Prove formula (1.2.6) by using the duplication formula for gamma functions.

1.2.4. Show that

$$\psi(1+n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma\ .$$

1.2.5. Evaluate $\psi(-\frac{3}{2})$.

1.2.6. Evaluate $\psi(5)$.

1.2.7. If $\ln\Gamma(z+1) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$ evaluate a_n, $n = 0, 1, 2, \cdots$

1.2.8. Show that $\zeta(k, \frac{1}{2}) = (2^k - 1)\zeta(k)$.

1.2.9. show that $\zeta(k, -\frac{3}{2}) = (-1)^k\Big(2^k\Big)\Big[1 + \frac{1}{3^k}\Big] + \zeta(k, \frac{1}{2})$.

1.2.10. Show that

$$\zeta\Big(k, z - \frac{2r+1}{2}\Big) = \frac{1}{\Big(z - \frac{1}{2}\Big)^k} + \cdots + \frac{1}{\Big(z - \frac{2r+1}{2}\Big)^k} + \zeta\Big(k, z + \frac{1}{2}\Big), r = 0, 1, \cdots, \quad k = 2, 3, \cdots$$

1.3. Integral Transforms

Basic integral transforms are the Mellin transform, the Laplace transform and the Fourier transform. Once the transforms are given, the unique functions which are recovered from these transforms are known as the inverse transforms such as inverse Mellin transform, inverse Laplace transform and inverse Fourier transform respectively. Depending upon the kernel function in the integral transform we have many other transforms such as the Bessel transform, Whittaker transform, Hankel transform, Stieltjes transform, Laguerre transform, hypergeometric transform, K-transform, Y-transform, G-transform, H-transform etc.

1.3.1. Mellin transform

Notation 1.3.1. $M_f(s)$: Mellin transform of $f(x)$ with parameter s

Definition 1.3.1. The Mellin transform of a real scalar function $f(x)$ with parameter s is defined as

$$M_f(s) = \int_0^\infty x^{s-1} f(x)\mathrm{d}x \tag{1.3.1}$$

whenever $M_f(s)$ exists.

It is a function of the arbitrary parameter s. Existence conditions for the Mellin and inverse Mellin transforms are available from books on complex analysis. Some detailed conditions are given in Mathai (1993). Since the participants of the School are a mixed group it is unwise to go into the theory of integral transforms and the details of existence conditions. We will introduce the basic transforms and illustrate how to use these transforms to solve practical problems.

Example 1.3.1. Evaluate the Mellin transform of the function $f(x) = e^{-x}$ for $x > 0$.

Solution 1.3.1.

$$M_f(s) = \int_0^\infty x^{s-1}e^{-x}dx = \Gamma(s) \text{ for } \Re(s) > 0 \tag{1.3.2}$$

from the integral representation of the gamma function in (1.1.6).

Thus given $M_f(s) = \Gamma(s)$ what is that function which gives rise to this $M_f(s)$. We know that one such function, if there exists many functions, is e^{-x}. Under the conditions of uniqueness for the existence of inverse, the inverse function is uniquely determined as e^{-x}. The formula for the inverse Mellin transform is the following.

$$f(x) = \frac{1}{2\pi i}\int_L M_f(s)x^{-s}ds, i = \sqrt{-1} \tag{1.3.3}$$

and L is a suitable contour, usually $L = \{c - i\infty, c + i\infty\}$ for some real c. Let us evaluate the inverse transform for $M_f(s)$ in Example 1.3.1.

Example 1.3.2. Given $M_f(s) = \Gamma(s)$ evaluate $f(x)$.

Solution 1.3.2. By the equation (1.3.3), $f(x)$ is given by the formula

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)x^{-s}ds. \tag{1.3.4}$$

This is a contour integral or an integral in the complex domain. The poles of the integrand $\Gamma(s)x^{-s}$ are coming from the poles of $\Gamma(s)$, which are at the points $s = 0, -1, -2, \cdots$. (See Definition 1.1.1). By the residue theorem in complex analysis, $f(x)$ in (1.3.3) is available as the sum of the residues of the integrand at the poles $s = 0, -1, -2, \cdots$. The residue at $s = -\nu$, denoted by $\Re_\nu$ is given by

$$\Re_\nu = \lim_{s\to-\nu}(s + \nu)[\Gamma(s)x^{-s}].$$

Since direct substitution will give an indeterminate quantity we may seek help from the property of gamma function in (1.1.10). That is,

$$(s+\nu)\Gamma(s) = (s+\nu)\frac{(s+\nu-1)\cdots s\Gamma(s)}{(s+\nu-1)\cdots s} = \frac{\Gamma(s+\nu+1)}{(s+\nu-1)\cdots s}.$$

Now, direct substitution is possible and hence

$$\mathfrak{R}_\nu = \lim_{s\to-\nu}\frac{\Gamma(s+\nu+1)x^{-s}}{(s+\nu-1)\cdots s} = \frac{\Gamma(1)x^\nu}{(-1)(-2)\cdots(-\nu)} = \frac{(-1)^\nu}{\nu!}x^\nu.$$

Hence the sum of the residues is given by

$$\sum_{\nu=0}^{\infty}\mathfrak{R}(\nu) = \sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}x^\nu = e^{-x}$$

which is the inverse function recovered from $M_f(s) = \Gamma(s)$. We may also observe from (1.3.4) that the poles are at $s = 0, -1, -2, \cdots$ and hence if a straight line contour $c - i\infty$ to $c + i\infty$ is taken with any $c > 0$ then all the poles of the integrand in (1.3.4) lie to the left of the contour. Then an infinite semi-circle can enclose all these poles and the residue theorem applies immediately.

Definition 1.3.2. Residue at $z = a$. If the scalar function $\phi(z)$ in the complex domain has a pole of order m at $z = a$ then the residue at $z = a$, denoted by R_a is given by the following:

$$\begin{aligned} R_a &= \lim_{z\to a}\left\{\frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}(z-a)^m\phi(z)\right]\right\} \qquad (1.3.5)\\ &= \lim_{z\to a}[(z-a)\phi(z)] \quad \text{for} \quad m=1 \quad \text{or for a simple pole at} \quad z=a. \end{aligned}$$

For example,

$$\phi_1(z) = \frac{z^5}{(z-1)(z-3)}$$

has simple poles or poles of order 1 at $z = 1$ and at $z = 3$, whereas

$$\phi_2(z) = \frac{e^{-z}}{(z-2)^3(z+1)}$$

has a simple pole at $z = -1$ and a pole of order 3 at $z = 2$. The residue of $\phi_1(z)$ at $z = 1$ is then

$$R_1 = \lim_{z\to 1} \frac{(z-1)z^5}{(z-1)(z-3)} = \lim_{z\to 1} \frac{z^5}{z-3} = \frac{1^5}{1-3} = -\frac{1}{2}$$

and the residue of $\phi_1(z)$ at $z = 3$ is given by,

$$R_3 = \lim_{z\to 3} \frac{(z-3)z^5}{(z-1)(z-3)} = \lim_{z\to 3} \frac{z^5}{z-1} = \frac{3^5}{3-1} = \frac{3^5}{2}.$$

The residue at $z = 2$ in $\phi_2(z)$ is given by the following:

$$\begin{aligned}
R_2 &= \lim_{z\to 2} \left\{ \frac{1}{2!} \left[\frac{d^2}{dz^2} (z-2)^3 \frac{e^{-z}}{(z-2)^3(z+1)} \right] \right\} \\
&= \lim_{z\to 2} \left\{ \frac{1}{2} \left[\frac{d^2}{dz^2} \frac{e^{-z}}{z+1} \right] \right\} \\
&= \lim_{z\to 2} \left\{ \frac{1}{2} \left[e^{-z} \left(\frac{1}{z+1} + \frac{2}{(z+1)^2} + \frac{2}{(z+1)^3} \right) \right] \right\} \\
&= \frac{1}{2} e^{-2} \left(\frac{1}{3} + \frac{2}{9} + \frac{2}{27} \right).
\end{aligned}$$

1.3.2. Laplace transform

Notation 1.3.2. $L_f(t)$: Laplace transform of f with parameter t.

Definition 1.3.3. The Laplace transform of a real scalar function $f(x)$ of the real variable x with parameter t is defined as

$$L_f(t) = \int_0^\infty e^{-tx} f(x) dx \tag{1.3.6}$$

whenever $L_f(t)$ exists. The inverse Laplace transform is given by

$$f(x) = \frac{1}{2\pi i} \int_L L_f(t) e^{tx} dt, \quad i = \sqrt{-1} \tag{1.3.7}$$

where L is a suitable contour. Thus (1.3.6) and (1.3.7) are known as the Laplace inverse Laplace pair.

Example 1.3.3. Evaluate the Laplace transform of the following

$$f(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}\mathrm{e}^{-x}, \quad x > 0, \Re(\alpha) > 0 \text{ and } f(x) = 0 \text{ elsewhere}.$$

Solution 1.3.3.

$$\begin{aligned} L_f(t) &= \int_0^\infty \mathrm{e}^{-tx}\frac{x^{\alpha-1}\mathrm{e}^{-x}}{\Gamma(\alpha)}\mathrm{d}x = \int_0^\infty \frac{x^{\alpha-1}}{\Gamma(\alpha)}\mathrm{e}^{-(1+t)x}\mathrm{d}x \\ &= (1+t)^{-\alpha} \text{ for } 1+t > 0. \end{aligned}$$

The Laplace transform in (1.3.6) need not exist always. But if t is replaced by it $i = \sqrt{-1}$, then

$$\mathrm{e}^{-itx} = \cos tx - i\sin tx \text{ and } |\mathrm{e}^{-itx}| = |\cos tx - i\sin tx| = 1.$$

Hence

$$\Big|\int_{-\infty}^\infty \mathrm{e}^{-itx} f(x)\mathrm{d}x\Big| \le \int_{-\infty}^\infty |f(x)|\mathrm{d}x.$$

Therefore, if $f(x)$ is an absolutely integrable function in the sense $\int_{-\infty}^\infty |f(x)|\mathrm{d}x < \infty$ then $\int_{-\infty}^\infty \mathrm{e}^{-itx} f(x)\mathrm{d}x$ always exists. This is known as the Fourier transform of $f(x)$, denoted by $F_f(t)$. That is,

$$F_f(t) = \int_{-\infty}^\infty \mathrm{e}^{-itx} f(x)\mathrm{d}x, \quad i = \sqrt{-1}. \tag{1.3.8}$$

More aspects of the basic transform will be discussed after introducing statistical densities.

Exercises 1.3.

1.3.1. Convolution property for Mellin transform. Let

$$g(u) = \int_0^\infty \frac{1}{v} f_1(v) f_2(\frac{u}{v})\mathrm{d}v$$

Then show that the Mellin transform of $g(u)$ with parameter s, denoted by $h(s)$ is the product of the Mellin transforms of $f_1(x)$ and $f_2(y)$ respectively. That is, $h(s) = h_1(s)h_2(s)$, $h_1(s) = \int_0^\infty x^{s-1} f_1(x)\mathrm{d}x$, $h_2(s) = \int_0^\infty y^{s-1} f_2(y)\,\mathrm{d}y$.

1.3.2. Show that

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x = \int_0^1 y^{\beta-1}(1-y)^{\alpha-1}\mathrm{d}y = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \Re(\alpha) > 0, \Re(\beta) > 0.$$

1.3.3. Show that

$$\int_0^\infty x^{\alpha-1}(1+x)^{-(\alpha+\beta)}\mathrm{d}x = \int_0^\infty y^{\beta-1}(1+y)^{-(\alpha+\beta)}\mathrm{d}y = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \Re(\alpha) > 0, \Re(\beta) > 0.$$

1.3.4. By using Exercise 1.3.2. or otherwise, evaluate the Mellin transform of the function

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \le x \le 1,$$
$$\Re(\alpha) > 0, \Re(\beta) > 0 \text{ and } f(x) = 0 \text{ elsewhere}.$$

1.3.5. By using Exercise 1.3.3. or otherwise, evaluate the Mellin transform of the function

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1+x)^{-(\alpha+\beta)},$$
$$x > 0, \Re(\alpha) > 0, \Re(\beta) > 0 \text{ and } f(x) = 0 \text{ elsewhere}.$$

1.3.6. Evaluate the Laplace transform of the function in Exercise 1.3.4. if it exists.

1.3.7. Evaluate the Laplace transform of the function in Exercise 1.3.5. if it exists.

1.3.8. Convolution property of Laplace transforms. Let

$$g(u) = \int_0^u f_1(u-x) f_2(x)\mathrm{d}x$$

Then the Laplace transform of $g(u)$ is the product of the Laplace transform of $f_1(x)$ and $f_2(x)$ respectively, whenever the Laplace transforms exist.

1.3.9. Let $f(x)$ be a real scalar function of the real and positive variable x. Consider the transformation $y = -\ln x$. Then show that the Mellin transform of $f(x)$ with parameter s is the same as the Laplace transform of the corresponding function of y with parameter s when the transform exist.

1.3.10. Evaluate the inverse Laplace transform of

$$L_f(t) = \frac{1}{(1+2t)(1+3t)}.$$

1.4. Some Statistical Preliminaries

A *random experiment* is an experiment where the outcomes are *not deterministic*. If the purpose of an experiment is to see whether a gold coin will sink in water then the outcome is predetermined. The coin will sink in water. It is not a random experiment. Consider an experiment of throwing a coin. Call one side of the coin "head" and the other side "tail". If the aim is to see whether head or tail will turn up when the coin is thrown once then the outcome is not predetermined. There is a chance that the head may turn up. There is also a chance that the tail may be the one turning up. This is a random experiment. If we assume that the coin will not stand on its edge and that it will fall head (H) or tail (T) for sure then there are two possible outcomes in this random experiment. The outcome set, called *"sample space"*, is then $\{H, T\}$. If a die (a cube with six faces marked $1, 2, 3, 4, 5, 6$) is rolled once then either 1 may turn up or 2 may turn up, $\cdots$, or 6 may turn up. The sample space here is $\{1, 2, 3, 4, 5, 6\}$.

An *event* is a subset of the sample space. In the case of the die the event of rolling a number between 3 and 5 (inclusive) is the set $\{3, 4, 5\}$. The event of rolling an even number is the set $\{2, 4, 6\}$. In the case of throwing a coin once the event of getting a head $A = \{H\}$ and the event of getting a tail $B = \{T\}$. The "chance" of the occurrence of the event A is called the probability of A, denoted by $P_r(A)$ or simply $P(A)$. Let us assign a number between 0 and 1 to measure $P(A)$. In the experiment of throwing a coin once the event of getting two heads is impossible or it is a *null set*, denoted by O. The event of getting either a head or tail when the coin is thrown once is sure to happen. It is a *sure event*, denoted by S. Let us assign the number zero to the *impossible event* O and the number one to the sure event S. Then any event $C \subset S$ (subset of S) has the probability $0 \leq P(C) \leq 1$. In the random experiment of throwing a coin once the events $A = \{H\}$ and $B = \{T\}$ are *mutually exclusive* because when A occurs B cannot occur or their intersection is null or $A \cap B = O$. In this experiment the union of these two events is the sure event itself, that is $A \cup B = S$. In the experiment of rolling a die once consider the following events: $A_1 = \{1\}, A_2 = \{3, 5\}, A_3 = \{4\}, A_4 = \{2, 6\}$. Then $A_i \cap A_j = O$ for all $i \neq j$ and $A_1 \cup A_2 \cup A_3 \cup A_4 \cup = S$. These events are then called *mutually exclusive* (intersections are null sets) and *totally exhaustive* (union is the sure event). The probability of an event will be defined by using the following postulates. For any sample space S of a random experiment let A be an event, $A \subset S$, with probability of A denoted by $P(A)$. Then $P(A)$ is assumed to satisfy the following postulates:

$$\begin{aligned} &\text{(i)} \quad 0 \leq P(A) \leq 1 \\ &\text{(ii)} \quad P(S) = 1 \\ &\text{(iii)} \quad P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots \quad \text{whenever} \quad A_1, A_2, \cdots \end{aligned} \tag{1.4.1}$$

are mutually exclusive.

The above postulates will not help to evaluate the probability of an event in a given situation. What is the probability that it will rain at 12 noon tomorrow over this lecture hall? There are two possibilities: A = event that it will rain, B = event that it will not rain. These are mutually exclusive and totally exhaustive and hence

$$1 = P(S) = P(A \cup B) = P(A) + P(B)$$

from postulates (ii) and (iii) and we know that $0 \leq P(A) \leq 1$. Since there are only two possibilities A and B we cannot conclude that $P(A) = \frac{1}{2}$. These two events obviously do not have equal probabilities. A meteorologist will be able to give a good estimate for $P(A)$.

In the case of throwing a coin once what is the probability of getting a head? If the events are $A = \{H\}$ and $B = \{T\}$ then as before we can come to the equation

$$1 = P(A) + P(B) \quad \text{with} \quad 0 \leq P(A) \leq 1.$$

We cannot say that $P(A) = \frac{1}{2}$ claiming that there are only two possibilities. In the case of rain we have seen that possibilities cannot be assigned by simply looking at the number of possibilities. If there is no way of preferring one side to the other or if there is symmetry in the outcomes of the random experiment then one way is to assign equal probabilities. If there is symmetry with respect to all characteristics of the coin then we say that the coin is *balanced*. In this case we will assign equal probabilities $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{2}$.

Example 1.4.1. If a “balanced” coin is tossed twice the possibilities are $\{HH\}$, $\{HT\}$, $\{TH\}$, $\{TT\}$ where the first letter denotes the outcome in the first trial. Since we assumed symmetry in the outcomes we assign equal probabilities to the events $P(A_1) = \frac{1}{4}$, $P(A_2) = \frac{1}{4}$, $P(A_3) = \frac{1}{4}$, $P(A_4) = \frac{1}{4}$ where $A_1 = \{H, H\}$, $A_2 = \{H, T\}$, $A_3 = \{T, H\}$, $A_4 = \{T, T\}$. Let x denote the number of heads. Then x can take the values $2, 1, 0$. Thus x is a variable. Further, we can assign probabilities to the values x takes. Probability that $x = 2$ is the probability of the event A_1. But $x = 1$ means either A_2 or A_3 has occurred with probabilities $\frac{1}{4}$ each. Hence we have the following *probability function* for x , denoted by $f(x)$

$$f(x) = \begin{cases} \frac{1}{4}, & \text{for } x = 2 \\ \frac{2}{4}, & \text{for } x = 1 \\ \frac{1}{4}, & \text{for } x = 0 \\ 0, & \text{elsewhere.} \end{cases}$$

This is an example of a *discrete random variable*, discrete in the sense of taking individually distinct values, such as $2, 1, 0$, with nonzero probabilities. Observe also that we can define a function of the following type $F(y) =$ probability that x is less than or equal to y, or written as

$$F(y) = Pr\{x \le y\}, \text{cumulative probabilities up to } y$$

for all real values of y. In our example above, $F(y) = 0$ for all y such that $-\infty < y < 0$. But at $y = 0$ there is a probability $\frac{1}{4}$ and there is no probability for the interval $0 < y < 1$. But at $y = 1$ there is another probability of $\frac{1}{2}$ and no probability between $1 < y < 2$ and then $\frac{1}{4}$ at $y = 2$. Thus $F(y)$ is a step function of the following form:

$$f(x) = \begin{cases} 0, & -\infty < y < 0 \\ \frac{1}{4}, & 0 \le y < 1 \\ \frac{3}{4}, & 1 \le y < 2 \\ 1, & 2 \le y < \infty. \end{cases}$$

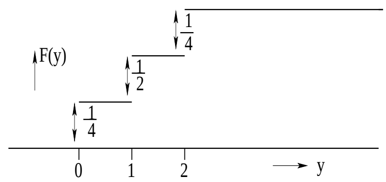

Figure 1.4.1

Example 1.4.2. Random cut. A child played with scissors and cut a string of length l into two pieces. Let one end of the uncut string be denoted by P , the other end by Q and the point of cut by C. Let the distance of PC be denoted by x. Note that x is a variable and we can also make probability statement on x. A convenient way to assign probabilities to x is to assign "relative length" as probabilities. What is the probability that the cut C is between 8.2 and 9.4, where $l = 10$.

Figure 1.4.1

Then

$$Pr\{8.2 \le x \le 9.4\} = \frac{9.4 - 8.2}{10} = \frac{1.2}{10} = 0.12$$

is the probability that x falls between 8.2 and 9.4 or that C falls between 8.2 and 9.4. What is the probability that the cut is between 11 and 11.5. This, of course is zero because it is an impossible event. What is the probability that C is on an interval of length Δx over the closed interval $[0, 10]$? The answer is obviously $\frac{\Delta x}{10}$. Then we may associate a function $f(x)$ with this random variable x that

$$f(x) = \begin{cases} \frac{1}{10}, & 0 \le x \le 10 \\ 0, & elsewhere. \end{cases}$$

This x is defined on a continuum of of points with nonzero probabilities and hence it is called a *continuous random variable* as opposed to a discrete random variable. In this case what is the probability that $x = 2.3$? The length is zero here and hence

$$Pr\{x = 2.3\} = \frac{2.3 - 2.3}{10} = 0.$$

In this example also one can look at the cumulative probability function.
Let

$$F(y) = Pr\{x \le y\}$$
$$= \begin{cases} 0, & -\infty < y < 0 \\ \int_0^y \frac{1}{10} dx = \frac{y}{10}, & 0 \le y \le 10 \\ 1, & y \ge 10. \end{cases}$$

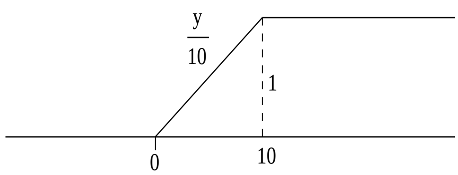

Figure 1.4.3

Definition 1.4.1. A random variable. A real variable x for which a probability statement of the type $Pr\{x \le y\}$ makes sense for all real $y, -\infty < y < \infty$, is called a *real random variable*.

If x takes individually distinct values with nonzero probabilities (as in Example 1.4.1) then x is called a *discrete random variable* whereas if x takes a continuum of points with nonzero probabilities (as in Example 1.4.2) then x is called a *continuous random variable*.

Definition 1.4.2. The *distribution function* or *cumulative probability function*. For any random variable x, $F(y) = Pr\{x \leq y\}$ is called the *distribution function* associated with x.

If x is discrete then $F(y)$ will be a step function. In general, whether x is discrete or continuous or mixed, $F(y)$ satisfies the following conditions.

$$\begin{aligned} &\text{(i)} \quad F(-\infty) = 0 \\ &\text{(ii)} \quad F(\infty) = 1 \\ &\text{(iii)} \quad F(a) \leq F(b) \text{ for } a < b. \end{aligned} \tag{1.4.2}$$

If $F(x)$ corresponds to a continuous random variable and if $F(x)$ is differentiable then

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x}F(x) \tag{1.4.3}$$

is the *density-function* for the continuous random variable x. $F(x)$ is a step function then the *probability-function* $f(x)$ of the discrete random variable x is available by taking successive differences. A density or probability function satisfies the following conditions:

$$\text{(i)} \quad f(x) \geq 0 \text{ for all } x \tag{1.4.4}$$

$$\text{(ii)} \quad \int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1 \text{ if } x \text{ is continuous and } \sum_{-\infty}^{\infty} f(x) = 1 \text{ if } x \text{ is discrete.}$$

Example 1.4.3. Check whether the following are density functions corresponding to a continuous real random variable x.

(a) $f(x) = \lambda e^{-\lambda x}, x > 0, \lambda > 0$ and $f(x) = 0$ elsewhere.

(b) $f(x) = \dfrac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}, \quad x > 0, \alpha > 0, \beta > 0$ and $f(x) = 0$ elsewhere.

(c) $f(x) = \dfrac{1}{b-a}, \quad a \le x \le b, a < b$ and $f(x) = 0$ elsewhere.

Solution 1.4.1.

(a) Is $f(x) \ge 0$ for all x ? Obviously it is true. Is the total integral 1?

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 0 + \int_{0}^{\infty} \lambda e^{-\lambda x}\mathrm{d}x = \int_{0}^{\infty} e^{-y}\mathrm{d}y, \ (y = \lambda x)$$
$$= 1.$$

Hence it is a density function. This density is known as the exponential density. What is the probability that x takes values between 2 and 5.7 in this case?

$$Pr\{2 \le x \le 5.7\} = \int_{2}^{5.7} \lambda e^{-\lambda x}\mathrm{d}x$$
$$= \text{area under the curve between the ordinates at } x = 2 \text{ and } x = 5.7.$$

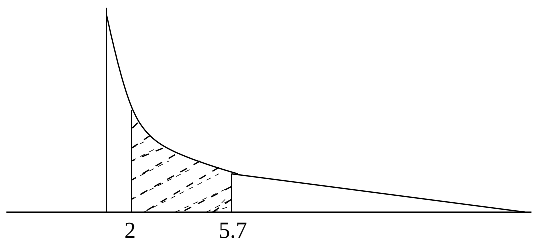

Figure 1.4.4

In the continuous case, in general, the probabilities are the areas under the curve and the total area = total probability = 1.

(b) It is obvious that $f(x) \ge 0$ for all x. Is the total integral 1?

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 0 + \int_0^{\infty} \frac{x^{\alpha-1}\mathrm{e}^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}\mathrm{d}x = \int_0^{\infty} \frac{y^{\alpha-1}\mathrm{e}^{-y}}{\Gamma(\alpha)}\mathrm{d}y, \quad \left(y = \frac{x}{\beta}\right)$$
$$= \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1.$$

It is a density. This is called a *gamma density* or a *two parameter gamma density*. It is a density whatever be the values of $\alpha > 0$ and $\beta > 0$. Such unknowns in a density function are called *parameters*. Here there are two parameters and in the example (a) above there was one parameter λ.

(c) The first condition is obvious and hence we check the second condition.

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 0 + \int_a^b \frac{1}{b-a}\mathrm{d}x = \left[\frac{x}{b-a}\right]_a^b = \frac{b-a}{b-a} = 1.$$

This is called a *uniform density* in the sense that the probability is uniformly distributed over the closed interval $[a, b]$ in the sense that the probabilities, which are areas under the curve, over intervals of equal lengths are equal wherever be the intervals taken from $[a, b]$.

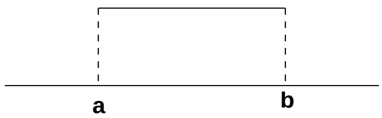

Figure 1.4.5

Example 1.4.4. Check whether the following are probability functions corresponding to some discrete random variable x.

(a) $f(x) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, 2, \cdots, n,\ 0 < p < 1$ and $f(x) = 0$ elsewhere.

(b) $f(x) = \frac{\lambda^x}{x!}\mathrm{e}^{-\lambda}, x = 0, 1, 2, \cdots,\ \lambda > 0$ and $f(x) = 0$ elsewhere.

Solution 1.4.2.

(a) The first condition is obvious. Hence we check the second condition.

$$\sum_{-\infty}^{\infty} f(x) = 0 + \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = [p + (1-p)]^n = 1^n = 1.$$

Hence it is a probability function. It is called the *binomial probability function* and x here is called a *binomial random variable* because the probability function $f(x)$ is the general term in the binomial expansion $[a+b]^n$ where $a = p, b = 1-p$.

(b) The first condition is obvious. Consider

$$\sum_{-\infty}^{\infty} f(x) = 0 + \sum_{x=0}^{n} \frac{\lambda^x}{x!} e^{-\lambda} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1.$$

This is known as the *Poisson probability function*, named after its inventor S. Poisson, a French mathematician. Note that we have only probability masses at individual points $x = 0, x = 1, \cdots$. For example,

$$Pr\{x = 0\} = \frac{\lambda^0}{0!} e^{-\lambda} = e^{-\lambda}, \quad Pr\{x = 1\} = \frac{\lambda^1}{1!} e^{-\lambda} = \lambda e^{-\lambda},$$

and so on.

Example 1.4.5. Poisson arrivals. Consider an event taking place over time t, such as the arrival of telephone calls to a switchboard, arrival of customers at a check-out counter, arrival of cars for repair in a repair garrage, occurrence of earth quakes at a particular locality, and so on. Let the arrivals be governed by the following conditions: (i) The probability of an arrival in time interval t to $t + \Delta t$ is proportional to its length, say $\lambda \Delta t$. (ii) The probability of more than one arrival in this interval of length Δt is negligibly small, we take it as zero for all practical purposes. (iii) Arrival or non-arrival in $[t, t + \Delta t]$ has nothing to do with what happened before. Under these conditions, what is the probability of getting exactly x arrivals in time t. Let us denote this probability of x arrivals in time t by $f(x, t)$. Then $f(x, t + \Delta t)$ is the probability of x arrivals in time $t + \Delta t$ or in the interval $[0, t + \Delta t]$. This can happen in two ways: exactly x arrivals in $[0, t]$ and no arrivals in $[t, t + \Delta t]$ or exactly $x - 1$ arrivals in $[0, t]$ and one arrival in $[t, t + \Delta t]$. These are mutually exclusive events also. Hence,

$$f(x, t + \Delta t) = f(x, t)[1 - \lambda \Delta t] + f(x - 1, t)\lambda \Delta t. \tag{1.4.5}$$

That is,

$$\frac{f(x, t + \Delta t) - f(x, t)}{\Delta t} = \lambda[f(x - 1, t) - f(x, t)].$$

Taking the limit $\Delta t \to 0$ we have the difference - differential equation

$$\frac{\partial}{\partial t} f(x, t) = -\lambda[f(x, t) - f(x - 1, t)]. \tag{1.4.6}$$

This can be solved successively by noting that at $x = 0$, $f(x - 1, t) = 0$ since the number of arrivals has to be zero or more. Thus,

$$\frac{\partial}{\partial t} f(0, t) = -\lambda f(0, t) \Rightarrow f(0, t) = \mathrm{e}^{-\lambda t}.$$

Solving successively (the reader may also verify) we have,

$$f(x, t) = \frac{(\lambda t)^x}{x!} \mathrm{e}^{-\lambda t}, \lambda > 0, t > 0, x = 0, 1, 2, \cdots . \tag{1.4.7}$$

and zero elsewhere, or we have a Poisson probability-law with parameter λt.

Exercises 1.4.

1.4.1. Bernoulli probability-law. Show that $f(x) = p^x(1 - p)^{1-x}, x = 0, 1, 0 < p < 1$ and $f(x) = 0$ elsewhere is a probability function.

1.4.2. Discrete hypergeometric law. Show that

$$f(x) = \binom{a}{x}\binom{b}{n - x} \Big/ \binom{a + b}{n}, x = 0, 1, \cdots, n \text{ or } a;$$

b, a positive integers, is a probability function.

1.4.3. Bose-Einstein density. Show that

$$f(x) = \frac{1}{c[-1 + \exp(\alpha + \beta x)]}, 0 < x < \infty, \beta > 0$$

is a density where c is the *normalizing constant*. Evaluate c also.

1.4.4. Cauchy density. Show that

$$f(x) = \frac{c}{\Delta^2 + (x-\mu)^2}, -\infty < x < \infty, \Delta > 0$$

is a density where c is the normalizing constant. Evaluate c also.

1.4.5. Fermi-Dirac density. Show that

$$f(x) = \frac{1}{c[1+\exp(\alpha+\beta x)]}, 0 < x < \infty, \alpha \neq 0, \beta > 0,$$

is a density where c is the normalizing constant. Evaluate c also.

1.4.6. Generalized gamma (gamma, chisquare, exponential, Weibull, Rayleigh etc special cases). Show that

$$f(x) = cx^{\alpha-1}\mathrm{e}^{-ax^{\delta}}, \ \alpha > 0, a > 0, \delta > 0, \ 0 < x < \infty.$$

is a density function where c is the normalizing constant. Evaluate c also.

1.4.7. Helley's density. Show that

$$f(x) = \left(\frac{mg}{KT}\right)\mathrm{e}^{-(mg\,x)/(KT)}, \quad x > 0, m > 0, g > 0, T > 0$$

is a density function.

1.4.8. Helmert density. Show that

$$f(x) = \frac{n^{\frac{(n-1)}{2}}\,(\frac{x}{\sigma})^{n-2}\,\mathrm{e}^{-(\frac{nx^2}{2\sigma^2})}}{\sigma 2^{\frac{(n-3)}{2}}\Gamma(\frac{n-1}{2})}, \quad 0 < x < \infty, \sigma > 0,$$

n a positive integer, is a density function.

1.4.9. Normal or Gaussian density. Show that

$$f(x) = c\,\mathrm{e}^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}, -\infty < x < \infty, -\infty < \mu < \infty, \ \sigma > 0$$

is a density. Evaluate the normalizing constant c.

1.4.10. Maxwell-Boltzmann density. Show that

$$f(x) = \frac{4}{\sqrt{\pi}}\beta^{\frac{3}{2}}x^2\mathrm{e}^{-\beta x^2}, \ 0 < x < \infty, \beta > 0$$

is a density.

1.5. Some Properties of Random Variables

A few essential properties of random variables will be given here so that the relevance of special functions can be appreciated.

Notation 1.5.1. $E[\psi(x)]$: Expected value of $\psi(x)$

Definition 1.5.1.

$$E[\psi(x)] = \int_{-\infty}^{\infty} \psi(x) f(x) \mathrm{d}x \tag{1.5.1}$$

when x is continuous with density function $f(x)$

$$= \sum_{-\infty}^{\infty} \psi(x) f(x) \tag{1.5.2}$$

when x is discrete with probability function $f(x)$.

Example 1.5.1. Evaluate the expected value of x^r when x has an exponential density

$$f(x) = \lambda \mathrm{e}^{-\lambda x}, \quad x > 0, \lambda > 0 \text{ and } f(x) = 0 \text{ elsewhere.}$$

Solution 1.5.1.

$$E(x^r) = 0 + \int_0^{\infty} x^r \lambda \mathrm{e}^{-\lambda x} \mathrm{d}x = \lambda^{-r} \int_0^{\infty} y^{r+1-1} \mathrm{e}^{-y} \mathrm{d}y, y = \lambda x$$
$$= \lambda^{-r} \Gamma(r+1) = r! \lambda^{-r}$$

by evaluating with the help of gamma functions. For example, the expected values of x in this case is

$$E(x) = \frac{1}{\lambda}.$$

Example 1.5.2. Evaluate the expected value of a Poisson random variable x with parameter λ.

Solution 1.5.2.

$$E(x) = 0 + \sum_{x=0}^{\infty} x\frac{\lambda^x}{x!}\mathrm{e}^{-\lambda} = \mathrm{e}^{-\lambda}\sum_{x=1}^{\infty} x\frac{\lambda^x}{x!}$$
$$= \lambda\mathrm{e}^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!} = \lambda\mathrm{e}^{-\lambda}\mathrm{e}^{\lambda} = \lambda.$$

The expected value of the Poisson random variable is the parameter itself.

Definition 1.5.2. $\mu'_r = E(x^r)$ is called the r^{th} moment of x and $E(x-E(x))^r = \mu_r$ is called the r^{th} central moment of x.

$$\mu'_1 = E(x)$$

is also called the *mean value* of x or the *centre of gravity* in x. When x is discrete, taking values $x_1, \cdots, x_k$ with the corresponding probabilities $p_1, \cdots, p_k$, $p_i \ge 0$, $i = 1, \cdots, k$, $p_1 + \cdots + p_k = 1$ then

$$E(x) = \sum_{i=1}^{k} p_i x_i = \frac{\sum_{i=1}^{k} p_i x_i}{\sum_{i=1}^{k} p_i}. \tag{1.5.3}$$

This can be considered to be a physical system with weights $p_1, \cdots, p_k$ at $x_1, \cdots, x_k$ then $E(x)$ is the center of gravity of the system.

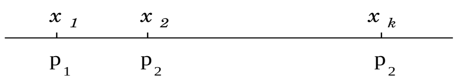

Figure 1.5.1

$$\mu_2 = E[x - E(x)]^2 = E[x^2 - xE(x) + [E(x)]^2]$$
$$= E(x^2) - [E(x)]^2 = \mathrm{Var}(x) \tag{1.5.4}$$

is called the *variance of* x and the positive square root $\sigma = \sqrt{\mathrm{Var}(x)}$ is called the *standard deviation* of x. Observe that σ can measure the spread or *disperson* in x from the point $E(x)$. In a physical system μ_2 can also represent the moment of inertia of the system. Observe that from the definition of expected value it follows that

$$E[a\psi(x) + b] = aE[\psi(x)] + b, \tag{1.5.5}$$

where a and b are constants.

$$E[e^{tx}] = M(t); \quad M(-t) = L_f(t), \tag{1.5.6}$$

is called the moment *generating function* of x. Observe that when t is replaced by $-t$ and when the variable is continuous with density function $f(x)$ for a positive random variable x then $M(-t)$ is the Laplace transform of $f(x)$. When t is replaced by $it, i = \sqrt{-1}$ we obtain the *characteristic function* of x and when it is replaced by $-it$ we obtain the Fourier transform of the density of x.

Example 1.5.3. Evaluate the variance of the random variable x having the density function

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{elsewhere.} \end{cases}$$

Solution 1.5.3.

$$\begin{aligned} E(x) &= 0 + \int_0^1 x(x)\mathrm{d}x + \int_1^2 x(2-x)\mathrm{d}x \\ &= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2 = 1. \\ E(x^2) &= 0 + \int_0^1 x^2(x)\mathrm{d}x + \int_1^2 x^2(2-x)\mathrm{d}x = \frac{7}{6}. \\ \mathrm{Var}(x) &= E(x^2) - [E(x)]^2 = \frac{7}{6} - 1^2 = \frac{1}{6}. \end{aligned}$$

Example 1.5.4. Evaluate the moment generating function for the Gaussian or normal density

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}, -\infty < x < \infty, \quad -\infty < \mu < \infty, \sigma > 0.$$

Solution 1.5.4.

$$\begin{aligned} M(t) &= E(\mathrm{e}^{tx}) = \mathrm{e}^{t\mu}E[\mathrm{e}^{t(x-\mu)}] \text{ since } \mathrm{e}^{t\mu} \text{is a constant} \\ &= \mathrm{e}^{t\mu}\int_{-\infty}^{\infty}\mathrm{e}^{t(x-\mu)}\frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}\mathrm{d}x. \end{aligned}$$

Put $y = \frac{x-\mu}{\sigma} \Rightarrow \mathrm{d}x = \sigma\mathrm{d}y$. Then

$$\begin{aligned} M(t) = \mathrm{e}^{t\mu} &= \int_{-\infty}^{\infty}\frac{\mathrm{e}^{t\sigma y-\frac{1}{2}y^2}}{\sqrt{2\pi}}\mathrm{d}y = \mathrm{e}^{t\mu+\frac{t^2\sigma^2}{2}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-z^2}}{\sqrt{\pi}}\mathrm{d}z. \\ &= \mathrm{e}^{t\mu+\frac{t^2\sigma^2}{2}}. \end{aligned}$$

The last part is evaluated with the help of a gamma function.

$$\begin{aligned} \int_{-\infty}^{\infty}\frac{\mathrm{e}^{-z^2}}{\sqrt{\pi}}\mathrm{d}z &= 2\int_0^{\infty}\frac{\mathrm{e}^{-z^2}}{\sqrt{\pi}}\mathrm{d}z \quad \text{due to evenness} \\ &= \int_0^{\infty}\frac{w^{\frac{1}{2}-1}\mathrm{e}^{-w}}{\sqrt{\pi}}\mathrm{d}w, w = z^2 \\ &= \frac{\Gamma(\frac{1}{2})}{\sqrt{\pi}} = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1. \end{aligned}$$

1.5.1. Multivariate analogues

A function $f(x_1, \cdots, x_k)$ of k real variables $(x_1, \cdots, x_k)$ is a probability function or a density function if it satisfies the following conditions:

(i) $f(x_1, \cdots, x_k) \geq 0$ for all $x_1, \cdots, x_k$

(ii) $\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1, \cdots, x_k)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k = 1$ if $x_1, \cdots, x_k$ are continuous and

$$\sum_{-\infty}^{\infty}\cdots\sum_{-\infty}^{\infty} f(x_1, \cdots, x_k) = 1 \text{ if } x_1, \cdots, x_k \text{ are discrete.} \tag{1.5.7}$$

For mixed cases when some variables are discrete and others continuous sum up the discrete ones and integrate the continuous ones.

One popular multivariate (many variables case) discrete situation is the multinomial probability law given by the probability function

$$f(x_1,\cdots,x_k)=\frac{n!}{x_1!\cdots x_k!}p_1^{x_1}\cdots p_k^{x_k}, p_i>0,\ i=1,\cdots,k,$$
$$p_1+\cdots+p_k=1,\quad x_i=0,1,\cdots,n,\quad i=1,\cdots k,\quad x_1+\cdots+x_k=n.$$

This is a $k-1$ variate probability function. Since $x_1+\cdots+x_k=n$ there are only $k-1$ free variables.

For any multivariate probability density function we can define expected values, product moments, joint moment generating function, joint characteristic function etc.

$$\begin{aligned} m(t_1,\cdots,t_k) &= E[\mathrm{e}^{t_1x_1+\cdots+t_kx_k}] \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{e}^{t_1x_1+\cdots+t_kx_k}f(x_1,\cdots,x_k)\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_k \end{aligned} \tag{1.5.8}$$

if $x_1,\cdots,x_k$ continuous with density $f(x_1,\cdots,x_k)$

$$= \sum_{-\infty}^{\infty}\cdots\sum_{-\infty}^{\infty}\mathrm{e}^{t_1x_1+\cdots+t_kx_k}f(x_1,\cdots,x_k) \tag{1.5.9}$$

if $x_1,\cdots,x_k$ are discrete, is the moment generating function of $x_1,\cdots,x_k$. If t_i is replaced by $-t_i$ for $i=1,\cdots,k$ then we obtain the Laplace transform of the density $f(x_1,\cdots,x_k)$ for $x_i>0,\ i=1,\cdots,k$. If t_j is replaced by $-it_j, i=\sqrt{-1}$ for $j=1,\cdots,k$ we have the Fourier transform of $f(x_1,\cdots,x_k)$ for x_j continuous for $j=1,\cdots,k$.

1.5.2. Marginal and conditional densities

If $f(x_1,\cdots,x_k)$ is the joint density of the random variables $x_1,\cdots,x_k$ then if we integrate out a few of the variables we obtain the joint *marginal density* of the remaining variables. For example

$$f_1(x_1)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x_1,x_2,x_3)\mathrm{d}x_2\wedge\mathrm{d}x_3 \tag{1.5.10}$$

is the marginal density of x_1 when $f(x_1,x_2,x_3)$ is the joint density of x_1,x_2 and x_3.

If $x_1,\cdots,x_k$ have the joint density $f(x_1,\cdots,x_k)$ and if $x_1,\cdots,x_r$ and $x_{r+1},\cdots,x_k$ have the joint marginal densities $g_1(x_1,\cdots,x_r)$ and $g_2(x_{r+1},\cdots,x_k)$ respectively then the conditional density of $x_1, \cdots, x_r$ given $x_{r+1} = a_{r+1}, \cdots, x_k = a_k$, where $a_1 \cdots, a_k$ are given numbers, is given by $h(x_1, \cdots, x_r | x_{r+1} = a_{r+1} \cdots, x_k = a_k)$

$$= \frac{f(x_1, \cdots, x_k)}{g_2(x_{r+1}, \cdots, x_k)} \text{ at } x_{k+1} = a_{k+1}, \cdots, x_k = a_k \tag{1.5.11}$$

provided $g_2(a_{k+1}, \cdots, a_k) \neq 0$.

Example 1.5.5. Evaluate the marginal densities of x_1 and x_2 and the conditional density of x_1 given $x_2 = \frac{1}{3}$ from the function

$$f(x_1, x_2) = x_1 + x_2, 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1 \text{ and } f(x_1, x_2) = 0 \text{ elsewhere,}$$

provided it is a joint density function. Check whether it is a joint density.

Solution 1.5.5. Let the marginal densities be denoted by $f_1(x_1)$ and $f_2(x_2)$ respectively.

$$f_1(x_1) = \int_{x_2} f(x_1, x_2)\mathrm{d}x_2 = \int_0^1 (x_1, x_2)\mathrm{d}x_2 = x_1 + \frac{1}{2}$$

$$f_2(x_2) = \int_{x_1} f(x_1, x_2)\mathrm{d}x_1 = \int_0^1 (x_1, x_2)\mathrm{d}x_1 = x_2 + \frac{1}{2}$$

$f(x_1, x_2) \geq 0$ for all x_1 and x_2. Further,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2)\mathrm{d}x_1 \wedge \mathrm{d}x_2 = 0 + \int_0^1\int_0^1 (x_1 + x_2)\mathrm{d}x_1 \wedge \mathrm{d}x_2$$
$$= \int_0^1 (x_1 + \frac{1}{2})\mathrm{d}x_1 = 1.$$

Hence $f(x_1, x_2)$ is a joint density and the marginal densities are as given above. The conditional density of x_1 given $x_2 = \frac{1}{3}$ is given by

$$h(x_1 | x_2 = \frac{1}{3}) = \frac{f(x_1, x_2)}{f_2(x_2)} \text{ at } x_2 = \frac{1}{3}$$
$$= \frac{x_1 + x_2}{x_2 + \frac{1}{2}}\Big|_{\frac{1}{3}} = \frac{x_1 + \frac{1}{3}}{\frac{1}{3} + \frac{1}{2}} = \frac{6}{5}(x_1 + \frac{1}{3}), 0 \leq x_1 \leq 1$$

and $h(x_1|x_2 = \frac{1}{3}) = 0$ elsewhere.

Observe that the notation for $x_1|x_2$ (x_1 given x_2) is a vertical bar after the first set of variables, and not a division symbol.

Exercises 1.5.

1.5.1. Evaluate the conditional density of y given x and (1) the conditional expectation of y given x, denoted by $E(y|x)$, (2) the conditional variance of y given x, denoted by $\mathrm{Var}(y|x)$ if the following is a joint density function of x and y. Verify that it is a joint density.

$$f(x,y) = \frac{1}{x^2\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(\frac{y-2-3x}{\sigma})^2}, -\infty < y < \infty,\quad 1 \le x < \infty$$

and $f(x,y) = 0$ elsewhere.

1.5.2. If the joint density is the product of the marginal densities then the random variables are said to be *independent* or *independently distributed*. Show that in (i) below the variables are independently distributed whereas in (ii) the variables are not independent.

(i) $f(x_1,x_2,x_3) = 6\mathrm{e}^{-x_1-2x_2-3x_3}, 0 \le x_1 < \infty, 0 \le x_2 < \infty, 0 \le x_3 < \infty$, and $f(x_1,x_2,x_3) = 0$ elsewhere.

(ii) $f(x,y) = x + y,\ 0 \le x \le 1,\ 0 \le y \le 1$ and zero elsewhere.

1.5.3. Evaluate the conditional density of x_1 given x_2 and x_3, denoted by $g_1(x_1|x_2,x_3)$ in (i) of Exercise 1.5.2 and the conditional density of x given y, denoted by $g_2(x|y)$ in (ii) of Exercise 1.5.2. Evaluate the conditional expectations $E(x_1|x_2 = 5, x_3 = 10), E(x|y = \frac{2}{3})$ and show that $E(x_1|x_2,x_3) = E(x_1)$ and $E(x|y) \neq E(x)$. [When the variables are independently distributed the conditional expectation is the same as the marginal expectation or it is free of the conditions imposed on the conditioned variables].

1.5.4. Let x_j have the gamma density

$$f_j(x_j) = \frac{x_j^{\alpha_j-1}}{\beta_j^{\alpha_j}\Gamma(\alpha_j)},\ \mathrm{e}^{-\frac{x_j}{\beta_j}}\ x_j \ge 0,\ \alpha_j > 0,\ \beta_j = \beta > 0$$

and $f_j(x_j) = 0$ elsewhere, for $j = 1, 2$. Assume that x_1 and x_2 are independently distributed [the joint density is the product of marginal densities when independent]. Consider $u = x_1 + x_2, v = \frac{x_1}{x_1+x_2}$ and $w = \frac{x_1}{x_2}$. Show that (i) u and v are independently distributed [Hint: Consider the transformation $x_1 = r\cos^2\theta, x_2 = r\sin^2\theta$], (ii) u is gamma distributed (u has a gamma density), and (iii) evaluate the densities of u and w.

1.5.5. Evaluate the joint moment generating function in Exercise 1.5.2 (i) and show that it is a product of the marginal (individual) moment generating functions due to independence of the variables.

1.5.6. Evaluate the joint moment generating function in Exercise 1.5.2 (ii) and show that it is not the product of the marginal (individual) moment generating functions.

1.5.7. The covariance between two random variables x and y, denoted by $\mathrm{Cov}(x, y)$ is defined for non-degenerate random variables ($\mathrm{Var}(x) \neq 0, \mathrm{Var}(y) \neq 0$). It is a measure of joint variation in (x, y) and it is defined as

$$\mathrm{Cov}(x, y) = E[x - E(x)][y - E(y)]$$

and the *linear correlation* in (x, y) is defined as $\rho = \frac{\mathrm{Cov}(x,y)}{\sqrt{\mathrm{Var}(x)\mathrm{Var}(y)}} = \rho(x, y)$. Show that whatever be the non-degenerate random variables x and y, $-1 \leq \rho \leq 1$ and $\rho = \pm 1$ if and only if $y = a + bx, b \neq 0, a, b$ constants.

1.5.8. Show that

(i) $\mathrm{Cov}(x, y) = E(xy) - E(x)E(y)$

(ii) $\rho(x, y) = \rho(ax + b, cy + d),\ a > 0, c > 0, a, b, c, d$ constants.

1.5.9. Evaluate $\mathrm{Cov}(x, y)$ and $\rho(x, y)$ in the joint density in Exercise 1.5.2 (ii)

1.5.10. Show that the following is a joint density of x and y:

$$f(x, y) = 2, 0 \leq x \leq y \leq 1 \ \text{ and zero elsewhere.}$$

For this joint density evaluate $\mathrm{Cov}(x, y)$ and $\rho(x, y)$.

1.6. Beta and Related Functions

Notation 1.6.1. $B(\alpha,\beta)$: **beta function**

Definition 1.6.1.

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \Re(\alpha) > 0, \Re(\beta) > 0. \tag{1.6.1}$$

One can give several types of integral representations for the beta function.

$$\begin{aligned} B(\alpha,\beta) &= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x,\ 0 \le x \le 1,\ \Re(\alpha) > 0, \Re(\beta) > 0, \qquad (1.6.2) \\ &= \int_0^1 x^{\beta-1}(1-x)^{\alpha-1}\mathrm{d}x. \end{aligned}$$

These are known as type-1 integral representations of a beta function. We can also show that

$$\begin{aligned} B(\alpha,\beta) &= \int_0^\infty x^{\alpha-1}(1+x)^{-(\alpha+\beta)}\mathrm{d}x \\ &= \int_0^\infty x^{\beta-1}(1+x)^{-(\alpha+\beta)}\mathrm{d}x, \quad \Re(\alpha) > 0, \Re(\beta) > 0. \qquad (1.6.3) \end{aligned}$$

These are known as type-2 integral representations of a beta function. The derivations of these integral representations can be done starting from the definition of a gamma function. Consider the integral representation

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x, \int_0^\infty y^{\beta-1}\mathrm{e}^{-y}\mathrm{d}y = \int_0^\infty \int_0^\infty x^{\alpha-1}y^{\beta-1}\mathrm{e}^{-(x+y)}\mathrm{d}x \wedge \mathrm{d}y$$

for $\Re(\alpha) > 0, \Re(\beta) > 0$. Make the transformation

$$x = r\cos^2\theta, y = r\sin^2\theta \Rightarrow \mathrm{d}x \wedge \mathrm{d}y = 2r\sin\theta\cos\theta\mathrm{d}r \wedge \mathrm{d}\theta.$$

Then

$$
\begin{aligned}
\Gamma(\alpha)\Gamma(\beta) &= \int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} r^{\alpha+\beta-1}\mathrm{e}^{-r}(\cos^2\theta)^{\alpha-1}(\sin^2\theta)^{\beta-1}2\sin\theta\cos\theta \mathrm{d}r\wedge\mathrm{d}\theta \\
&= \int_{r=0}^{\infty} r^{\alpha+\beta-1}\mathrm{e}^{-r}\mathrm{d}r\int_{\theta=0}^{\pi/2}(\cos^2\theta)^{\alpha-1}(\sin^2\theta)^{\beta-1}2\sin\theta\cos\theta\mathrm{d}\theta. \\
&= \Gamma(\alpha+\beta)\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\mathrm{d}u, \quad [u=\cos^2\theta \Rightarrow \mathrm{d}u = -2\cos\theta\sin\theta\mathrm{d}\theta].
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\mathrm{d}u &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \Re(\alpha)>0, \Re(\beta)>0. \qquad (1.6.4)\\
&= \int_0^1 v^{\beta-1}(1-v)^{\alpha-1}\mathrm{d}v, [v=1-u].
\end{aligned}
$$

Put

$$
w = \frac{u}{1-u} \Rightarrow \frac{1}{(1+w)^2}\mathrm{d}w = \mathrm{d}u, \quad u = \frac{w}{1+w}
$$

Then

$$
\begin{aligned}
\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\mathrm{d}u &= \int_0^{\infty} w^{\alpha-1}(1+w)^{-(\alpha+\beta)}\mathrm{d}w, \quad \Re(\alpha)>0, \Re(\beta)>0 \\
&= \int_0^{\infty} t^{\beta-1}(1+t)^{-(\alpha+\beta)}\mathrm{d}t, [t=\frac{1}{w}]. \qquad (1.6.5)
\end{aligned}
$$

With the help of type-1 and type-2 beta functions we can define the corresponding beta densities.

Definition 1.6.2. Type-1 beta density

$$
f_1(x) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}, \ 0 \le x \le 1, \quad \Re(\alpha)>0, \Re(\beta)>0 \qquad (1.6.6)
$$

and $f_1(x) = 0$ elsewhere.

Definition 1.6.3. Type-2 beta density

$$
f_2(x) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1+x)^{-(\alpha+\beta)}, \quad \Re(\alpha)>0, \Re(\beta)>0, x>0 \qquad (1.6.7)
$$

and $f_2(x) = 0$ elsewhere.

Note that both $f_1(x)$ and $f_2(x)$ satisfy non-negativity and the total integral being unity.

Example 1.6.1. Evaluate the h-th moment of x if x has

(i) a gamma distribution with density

$$f_1(x) = \frac{x^{\alpha-1}\mathrm{e}^{-x}}{\Gamma(\alpha)}, x \geq 0, \Re(\alpha) > 0, \Re(\beta) > 0, \text{ and } f_1(x) = 0 \text{ elsewhere;}$$

(ii) a type-1 beta distribution with density

$$f_2(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1},\ 0 \leq x \leq 1, \Re(\alpha) > 0, \text{ and } f_2(x) = 0 \text{ elsewhere;}$$

(iii) a type-2 beta density

$$f_3(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1+x)^{-(\alpha+\beta)},\ 0 \leq x \leq \infty, \Re(\alpha) > 0, \Re(\beta) > 0 \text{ and } f_3(x) = 0 \text{ elsewhere;}$$

Solution 1.6.1. (i)

$$\begin{aligned} E(x^h) &= \int_0^\infty x^h \frac{x^{\alpha-1}\mathrm{e}^{-x}}{\Gamma(\alpha)}\mathrm{d}x = \frac{1}{\Gamma(\alpha)}\int_0^\infty x^{\alpha+h-1}\mathrm{e}^{-x}\mathrm{d}x \\ &= \frac{\Gamma(\alpha+h)}{\Gamma(\alpha)} \text{ for } \Re(\alpha+h) > 0. \end{aligned} \tag{1.6.8}$$

Thus the $h-th$ moment exists for negative values of h also provided $\alpha+h > 0$ if α and h are real. In statistical problems usually the parameters are all real. For $h = s-1$ one has the Mellin transform of $f_1(x)$.

(ii)

$$\begin{aligned} E(x^h) &= \int_0^1 x^h \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{\alpha+h-1}(1-x)^{\beta-1}\mathrm{d}x \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+h)\Gamma(\beta)}{\Gamma(\alpha+\beta+h)} \text{ for } \Re(\alpha+h) > 0 \\ &= \frac{\Gamma(\alpha+h)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+h)}. \end{aligned} \tag{1.6.9}$$

For $h = s - 1$ one has the Mellin transform of $f_2(x)$. Thus, as an inverse Mellin transform, $f_2(x)$ is available from (1.6.9). That is,

$$f_2(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\alpha+s-1)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+s-1)}x^{-s}\mathrm{d}s,\quad i=\sqrt{-1}. \tag{1.6.10}$$

Evaluating this contour integral as the sum of the residues at the poles of $\Gamma(\alpha + s - 1)$ one obtains $f_2(x)$ as given in the example.

(iii)

$$\begin{aligned}
E(x^h) &= \int_0^\infty x^h \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1+x)^{-(\alpha+\beta)}\mathrm{d}x\\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty x^{\alpha+h-1}(1+x)^{-[(\alpha+h)+(\beta-h)]}\mathrm{d}x\\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+h)\Gamma(\beta-h)}{\Gamma(\alpha+\beta)}\ \text{ for }\ \Re(\alpha+h)>0, \Re(\beta-h)>0\\
&= \frac{\Gamma(\alpha+h)}{\Gamma(\alpha)}\frac{\Gamma(\beta-h)}{\Gamma(\beta)}\ \text{ for }\ -\Re(\alpha)<\Re(h)<\Re(\beta).
\end{aligned} \tag{1.6.11}$$

Thus, only a few moments satisfying the condition $-\alpha < h < \beta$ can exist when α and β are real.

1.6.1. Dirichlet Integrals and Dirichlet Densities

A multivariate integral, which is a generalization of a beta integral, is the Dirichlet integral. We looked at type-1 and type-2 beta integrals. Here we consider type-1 and type-2 Dirichlet integrals and their generalizations. Analogously we will also define the corresponding statistical densities.

Notation 1.6.2. **Dirichlet Function:** $D(\alpha_1, \cdots, \alpha_k; \alpha_{k+1})$ (real scalar case)

Definition 1.6.4.

$$D(\alpha_1,\cdots,\alpha_k;\alpha_{k+1}) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_{k+1})}{\Gamma(\alpha_1+\cdots+\alpha_{k+1})}\ \text{for}\ \Re(\alpha_j)>0,\ j=1,\cdots,k+1. \tag{1.6.12}$$

Note that for $k = 1$ we have the beta function in the real scalar case. Consider the following integral:

$$D_1 = \int_{\Omega} \cdots \int x_1^{\alpha_1 - 1}, \cdots, x_k^{\alpha_k - 1}(1 - x_1 - \cdots - x_k)^{\alpha_{k+1}-1} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k \quad (1.6.13)$$

where $\Omega = \{(x_1, \cdots, x_k) | 0 \leq x_i \leq 1, i = 1, \cdots, k, 0 \leq x_1 + \cdots + x_k \leq 1\}$. Since $1 - x_1 - \cdots - x_k \geq 0$ we have $0 \leq x_1 \leq 1 - x_2 - \cdots - x_k$. Integration over x_1 yields the following:

$$\begin{aligned} \int_{x_1=0}^{1-x_2-\cdots-x_k} & x_1^{\alpha_1-1}(1 - x_1 - x_2 - \cdots - x_k)^{\alpha_{k+1}-1} \mathrm{d}x_1 \\ &= (1 - x_2 - \cdots - x_k)^{\alpha_{k+1}-1} \\ &\times \int_{x_1=0}^{1-x_2-\cdots-x_k} x_1^{\alpha_1-1}\left[1 - \frac{x_1}{1 - x_2 - \cdots - x_k}\right]^{\alpha_{k+1}-1} \mathrm{d}x_1. \end{aligned}$$

Put

$$y_1 = \frac{x_1}{1 - x_2 - \cdots - x_k} \Rightarrow \mathrm{d}x_1 = (1 - x_2 - \cdots - x_k)\mathrm{d}y_1.$$

Then the integral over x_1 yields,

$$\begin{aligned} (1 - x_2 - \cdots - x_k)^{\alpha_1+\alpha_{k+1}-1} & \int_0^1 y_1^{\alpha_1-1}(1 - y_1)^{\alpha_{k+1}-1} \mathrm{d}y_1 \\ &= (1 - x_2 - \cdots - x_k)^{\alpha_1+\alpha_{k+1}-1} \frac{\Gamma(\alpha_1)\Gamma(\alpha_{k+1})}{\Gamma(\alpha_1 + \alpha_{k+1})} \end{aligned}$$

for $\Re(\alpha_1) > 0$, $\Re(\alpha_{k+1}) > 0$. Integral over x_2 yields,

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_{k+1})}{\Gamma(\alpha_1 + \alpha_{k+1})} \frac{\Gamma(\alpha_2)\Gamma(\alpha_1 + \alpha_{k+1})}{\Gamma(\alpha_1 + \alpha_2 + \alpha_{k+1})} = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_{k+1})}{\Gamma(\alpha_1 + \alpha_2 + \alpha_{k+1})}.$$

Proceeding like this, we have the final result:

$$D_1 = D(\alpha_1, \cdots, \alpha_k; \alpha_{k+1}) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2) \cdots \Gamma(\alpha_{k+1})}{\Gamma(\alpha_1 + \cdots + \alpha_{k+1})}, \ \Re(\alpha_j) > 0, j = 1, \cdots, k+1. \quad (1.6.14)$$

Here, (1.6.13) is the type-1 Dirichlet integral. Hence by normalizing the integrand in (1.6.13) we have the type-1 Dirichlet density.

Definition 1.6.5. **Type-1 Dirichlet density** $f_1(x_1, \cdots, x_k)$**.**

$$\begin{aligned} f_1(x_1, \cdots, x_k) &= \frac{1}{D(\alpha_1, \cdots, \alpha_k; \alpha_{k+1})} x_1^{\alpha_1 - 1} \cdots x_k^{\alpha_k - 1} (1 - x_1 - \cdots - x_k)^{\alpha_{k+1} - 1}, \\ &0 \le x_j \le 1,\, j = 1, \cdots, k,\; 0 \le x_1 + \cdots + x_k \le 1,\; \Re(\alpha_j) > 0, \\ &j = 1, \cdots, k+1, \text{ and } f_1(x_1, \cdots, x_k) = 0 \text{ elsewhere.} \end{aligned} \tag{1.6.15}$$

Consider the type-2 Dirichlet integral

$$D_2 = \int_0^\infty \cdots \int_0^\infty x_1^{\alpha_1 - 1} \cdots x_k^{\alpha_k - 1} (1 + x_1 + \cdots + x_k)^{-(\alpha_1 + \cdots + \alpha_{k+1})} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k. \tag{1.6.16}$$

This can be integrated by writing

$$(1 + x_1 + \cdots + x_k) = (1 + x_2 + \cdots + x_k)\left[1 + \frac{x_1}{1 + x_2 + \cdots + x_k}\right]$$

and then integrating out with the help of type-2 beta integrals. The final result will agree with the Dirichlet function

$$D_2 = D(\alpha_1, \cdots \alpha_k; \alpha_{k+1}). \tag{1.6.17}$$

Thus, we can define a type-2 Dirichlet density.

Definition 1.6.6. **Type-2 Dirichlet density**.

$$\begin{aligned} f_2(x_1, \cdots, x_k) &= \frac{1}{D(\alpha_1, \cdots, \alpha_k; \alpha_{k+1})} x_1^{\alpha_1 - 1} \cdots x_k^{\alpha_k - 1} (1 + x_1 + \cdots + x_k)^{-(\alpha_1 + \cdots + \alpha_{k+1})}, \\ &0 \le x_j < \infty,\; j = 1, \cdots, k,\; \Re(\alpha_j) > 0,\; j = 1, \cdots, k+1, \end{aligned} \tag{1.6.18}$$

and $f_2(x_1, \cdots, x_k) = 0$ elsewhere.

It is easy to observe that if $(x_1, \cdots, x_k)$ has a k - variate type-1 Dirichlet density then any subset of r of the variables have a r - variate type-1 Dirichlet density for $r = 1, \cdots, k$. Similarly if $(x_1, \cdots, x_k)$ have a type-2 Dirichlet density then any subset of them will have a type-2 Dirichlet density.

Example 1.6.2. Evaluate the marginal densities from the following bivariate density:

$$f(x_1, x_2) = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x_1^{\alpha_1-1} x_2^{\alpha_2-1}(1-x_1-x_2)^{\alpha_3-1}, 0 \le x_j \le 1, j = 1,2,3,$$
$$0 \le x_1+x_2+x_3 \le 1,\ \Re(\alpha_j) > 0, j = 1,2,3, \text{ and } f(x_1,x_2) = 0 \text{ elsewhere.}$$

Solution 1.6.2. Let the marginal densities be denoted by $f_1(x_1)$ and $f_2(x_2)$ respectively.

$$\begin{aligned} f_1(x_1) &= \int_{x_2} f(x_1x_2)\mathrm{d}x_2 = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x_1^{\alpha_1-1} \\ &\times \int_{x_2=0}^{1-x_1} x_2^{\alpha_2-1}(1-x_1-x_2)^{\alpha_3-1}\mathrm{d}x_2 \\ &= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x_1^{\alpha_1-1}(1-x_1)^{\alpha_3-1} \int_{x_2=0}^{1-x_1} x_2^{\alpha_2-1}\left[1-\frac{x_2}{1-x_1}\right]^{\alpha_3-1}\mathrm{d}x_2. \end{aligned}$$

Put

$$y_2 = \frac{x_2}{1-x_1} \Rightarrow \mathrm{d}x_2 = (1-x_1)\mathrm{d}y_2.$$
$$f_1(x_1) = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x_1^{\alpha_1-1}(1-x_1)^{\alpha_2+\alpha_3-1} \int_0^1 y_2^{\alpha_2-1}(1-y_2)^{\alpha_3-1}\mathrm{d}y_2.$$

Evaluating the y_2-integral with the help of a type-1 beta integral we obtain $\frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_2+\Gamma\alpha_3)}$. Hence,

$$f_1(x_1) = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2+\alpha_3)} x_1^{\alpha_1-1}(1-x_1)^{\alpha_2+\alpha_3-1}, 0 \le x_1 \le 1,$$

and zero elsewhere. From symmetry,

$$f_2(x_2) = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_2)\Gamma(\alpha_1+\alpha_3)} x_2^{\alpha_2-1}(1-x_2)^{\alpha_1+\alpha_3-1}\ 0 \le x_2 \le 1,$$

and zero elsewhere. Thus, the marginal densities of x_1 and x_2 are type-1 beta densities.

Example 1.6.3. Evaluate the normalizing constant c if the following is a density function:

$$f(x_1,x_2) = cx_1^{\alpha_1-1}(1-x_1)^{\beta_1}x_2^{\alpha_2-1}(1-x_1-x_2)^{\alpha_3-1}, 0 \le x_j \le 1, \tag{1.6.19}$$

$0 \le x_1 + x_2 \le 1$, $j = 1,2$, $\Re(\alpha_j) > 0$, $j = 1,2,3$ and $f(x_1,x_2) = 0$ elsewhere.

Solution 1.6.3. Let us integrate out x_2 first.

$$\begin{aligned}
&\int_{x_2=0}^{1-x_1} x_2^{\alpha_2-1}(1-x_1-x_2)^{\alpha_3-1}\mathrm{d}x_2 \\
&\qquad = (1-x_1)^{\alpha_2+\alpha_3-1}\int_0^1 y_2^{\alpha_2-1}(1-y_2)^{\alpha_3-1}\mathrm{d}y_2, y_2 = \frac{x_2}{1-x_1} \\
&\qquad = (1-x_1)^{\alpha_2+\alpha_3-1}\frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_2+\alpha_3)}, \quad \Re(\alpha_2) > 0, \quad \Re(\alpha_3) > 0.
\end{aligned}$$

Now, integrating out x_1 we have,

$$\int_0^1 x_1^{\alpha_1-1}(1-x_1)^{\beta_1+\alpha_2+\alpha_3-1}\mathrm{d}x_1 = \frac{\Gamma(\alpha_1)\Gamma(\beta_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\beta_1)}$$

$\Re(\alpha_1) > 0$, $\Re(\beta_1+\alpha_2+\alpha_3) > 0$. Hence

$$c = \frac{\Gamma(\alpha_2+\alpha_3)\Gamma(\alpha_1+\alpha_2+\alpha_3+\beta_1)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_2+\alpha_3+\beta_1)}, \quad \Re(\alpha_j) > 0, j = 1,2,3, \Re(\alpha_2+\alpha_3+\beta_1) > 0.$$

A generalization to k variable case is one of the generalizations of type-1 Dirichlet density and the corresponding type-1 Dirichlet function.

Example 1.6.4. Evaluate the normalizing constant if the following is a density function:

$$\begin{aligned}
f(x_1,x_2,x_3) &= cx_1^{\alpha_1-1}(x_1+x_2)^{\beta_2}\ x_2^{\alpha_2-1}(x_1+x_2+x_3)^{\beta_3}\ x_3^{\alpha_3-1}(1-x_1-x_2-x_3)^{\alpha_4-1}, \\
&0 \le x_1+\cdots+x_j \le 1, \quad j = 1,2,3,4, \quad \Re(\alpha_j) > 0, \quad j = 1,2,3,4, \quad (1.6.20) \\
&\Re(\alpha_1+\cdots+\alpha_j+\beta_2+\cdots+\beta_j) > 0, j = 1,2,3,4
\end{aligned}$$

and $f(x_1,x_2,x_3) = 0$ elsewhere.

Solution 1.6.4. Let $u_1 = x_1,\ u_2 = x_1 + x_2\ u_3 = x_1 + x_2 + x_3$ and let the joint density of u_1, u_2, u_3 be denoted by $g(u_1, u_2, u_3)$. Then

$$g(u_1,u_2,u_3) = c\ u_1^{\alpha_1-1}u_2^{\beta_2}(u_2-u_1)^{\alpha_2-1}u_3^{\beta_3}(u_3-u_2)^{\alpha_3-1}(1-u_3)^{\alpha_4-1},\\ 0 \le u_1 \le u_2 \le u_3 \le 1.$$

Note that $0 \le u_1 \le u_2$. Integration over u_1 yields the following:

$$\begin{aligned}\int_{u_1=0}^{u_2} u_1^{\alpha_1-1}(u_2-u_1)^{\alpha_2-1}\mathrm{d}u_1 &= u_2^{\alpha_2-1}\int_{u_1=0}^{u_2} u_1^{\alpha_1-1}(1-\frac{u_1}{u_2})^{\alpha_2-1}\mathrm{d}u_1\\ &= u_2^{\alpha_1+\alpha_2-1}\int_0^1 y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1}\mathrm{d}y_1,\ y_1 = \frac{u_1}{u_2}\\ &= u_2^{\alpha_1+\alpha_2-1}\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)},\ \Re(\alpha_1) > 0, \Re(\alpha_2) > 0.\end{aligned}$$

Integration over u_2 yields the following:

$$\int_{u_2=0}^{u_3} u_2^{\alpha_1+\alpha_2+\beta_2-1}(u_3-u_2)^{\alpha_3-1}\mathrm{d}u_2 = u_3^{\alpha_1+\alpha_2+\alpha_3+\beta_2-1}\frac{\Gamma(\alpha_3)\Gamma(\alpha_1+\alpha_2+\beta_2)}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\beta_2)}\\ \text{for } \Re(\alpha_3) > 0, \Re(\alpha_1+\alpha_2+\beta_2) > 0.$$

Finally, integral over u_3 yields the following:

$$\int_{u_3=0}^{1} u_3^{\alpha_1+\alpha_2+\alpha_3+\beta_2+\beta_3-1}(1-u_3)^{\alpha_4-1}\mathrm{d}u_3 = \frac{\Gamma(\alpha_4)\Gamma(\alpha_1+\alpha_2+\alpha_3+\beta_2+\beta_3)}{\Gamma(\alpha_1+\cdots+\alpha_4+\beta_2+\beta_3)},\\ \Re(\alpha_4) > 0, \Re(\alpha_1+\alpha_2+\alpha_3+\beta_2+\beta_3) > 0.$$

Hence

$$c^{-1} = \Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_4)\frac{\Gamma(\alpha_1+\alpha_2+\beta_2)}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\beta_2)}\\ \times\frac{\Gamma(\alpha_1+\alpha_2+\alpha_3+\beta_2+\beta_3)}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\beta_2+\beta_3)}$$

for $\Re(\alpha_j) > 0,\ j = 1,2,3,4,\ \Re(\alpha_1+\cdots+\alpha_j+\beta_2+\cdots+\beta_j) > 0,\ j = 2,3.$

Note that one can generalize the function in (1.8.9) to a k- variables situation. This will produce another generalization of the type-1 Dirichlet function as well as the type-1 Dirichlet density. Corresponding situations in the type-2 case will provide generalizations of the type-2 Dirichlet integral and density.

Exercises 1.6.

1.6.1. Let $f(x_1,x_2,x_3) = cx_1^{\alpha_1-1}(1+x_1)^{-(\alpha_1+\beta_1)}x_2^{\alpha_2-1}(1+x_1+x_2)^{-(\alpha_2+\beta_2)}x_3^{\alpha_3-1}(1+x_1+x_2+x_3)^{-(\alpha_3+\beta_3)},\ 0 \le x_j < \infty,\ j=1,2,3$ and $f(x_1,x_2,x_3)=0$ elsewhere. If $f(x_1,x_2,x_3)$ is a density function then evaluate c and write down the conditions on the parameters.

1.6.2. Generalize the density in Exercises 1.6.1 to k-variables case, evaluate the corresponding c and write down the conditions.

1.6.3. Write down the k-variables situation in Example 1.6.3 and evaluate the normalizing constant, and give the conditions on the parameters.

1.6.4. Write down the general density corresponding to Example 1.6.2 and evaluate the normalizing constant, and give the conditions on the parameters.

1.6.5. By using the gamma structure in the normalizing constant in Exercise 1.6.4 show that the joint density in Exercise 1.6.4 can also be obtained as the joint density of k mutually independently distributed real scalar type-1 beta random variables, and identify the parameters in these independent type-1 beta random variables.

1.7. Hypergeometric Series

A general hypergeometric series with p upper or numerator parameters and q lower or denominator parameters is denoted and defined as follows:

Notation 1.7.1.

$${}_pF_q\,(a_1\cdots a_p;b_1\cdots b_q;z) = {}_pF_q((a_p);(b_q);z) = {}_pF_q(z)$$

Definition 1.7.1.

$${}_pF_q(z) = \sum_{r=0}^{\infty}\frac{(a_1)_r\cdots(a_p)_r}{(b_1)_r\cdots(b_q)_r}\,\frac{z^r}{r!} \tag{1.7.1}$$

where $(a_j)_r$ and $(b_j)_r$ are the Pochammer symbols of (1.1.1). The series in (1.7.1) is defined when none of the b_j 's, $j = 1, \cdots q$, is a negative integer or zero. If a b_j is a negative integer or zero then $(b_j)_r$ will be zero for some r. A b_j can be zero provided there is a numerator parameter a_k such that $(a_k)_r$ becomes zero first before $(b_j)_r$ becomes zero. If any numerator parameter a_j is a negative integer or zero then (1.7.1) terminates and becomes a polynomial in z. From the ratio test it is evident that the series in (1.7.1) is convergent for all z if $q \geq p$, it is convergent for $|z| < 1$ if $p = q + 1$ and divergent if $p > q + 1$. When $p = q + 1$ and $|z| = 1$ the series can converge in some cases. Let

$$\beta = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j.$$

It can be shown that when $p = q + 1$ the series is absolutely convergent for $|z| = 1$ if $\Re(\beta) < 0$, conditionally convergent for $z = -1$ if $0 \leq \Re(\beta) < 1$ and divergent for $|z| = 1$ if $1 \leq \Re(\beta)$.

Some special cases of a ${}_pF_q$ are the following: When there is no upper or lower parameters we have,

$${}_0F_0(;;\pm z) = \sum_{r=0}^{\infty} \frac{(\pm z)^r}{r!} = e^{\pm z}. \tag{1.7.2}$$

Thus ${}_0F_0(.)$ is an exponential series.

$${}_1F_0(\alpha;;z) = \sum_{r=0}^{\infty} (\alpha)_r \frac{z^r}{r!} = (1-z)^{-\alpha} \text{ for } |z| < 1. \tag{1.7.3}$$

This is the binomial series. ${}_1F_1(.)$ is known as confluent hypergeometric series or *Kummers's hypergeometric series and* ${}_2F_1(.)$ is known as *Gauss' hypergeometric series* .

Example 1.7.1. Incomplete gamma function. Evaluate the incomplete gamma function

$$\gamma(\alpha, b) = \int_0^b x^{\alpha-1} e^{-x} dx, \; b < \infty$$

and write it in terms of a Kummer's hypergeometric function.

Solution 1.7.1. Since b is finite we may expand the exponential part and integrate term by term .

$$\gamma(\alpha, b) = \int_0^b x^{\alpha-1}\left\{\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}x^r\right\}\mathrm{d}x = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\int_0^b x^{\alpha+r-1}\mathrm{d}x$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{b^{\alpha+r}}{\alpha+r} = \frac{b^\alpha}{\alpha}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{(\alpha)_r}{(\alpha+1)_r}b^r$$

$$= \frac{b^\alpha}{\alpha}\ {}_1F_1(\alpha; \alpha+1; -b). \tag{1.7.4}$$

Hence the upper part

$$\Gamma(\alpha, b) = \int_b^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x = \Gamma(\alpha) - \gamma(\alpha, b). \tag{1.7.5}$$

Example 1.7.2. **Incomplete beta function**. Evaluate the incomplete beta function

$$b(\alpha, \beta; t) = \int_0^t x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x, t < 1$$

and write it in terms of a Gauss' hypergeometric function.

Solution 1.7.2. Note that since $0 < x < 1$,

$$(1-x)^{\beta-1} = (1-x)^{-(1-\beta)} = \sum_{r=0}^{\infty}\frac{(1-\beta)_r}{r!}x^r.$$

Hence,

$$b(\alpha, \beta; t) = \sum_{r=0}^{\infty}\frac{(1-\beta)_r}{r!}\int_0^t x^{\alpha+r-1}\mathrm{d}x = \sum_{r=0}^{\infty}\frac{(1-\beta)_r}{r!}\frac{t^{\alpha+r}}{\alpha+r}$$

$$= \frac{t^\alpha}{\alpha}\sum_{r=0}^{\infty}\frac{(1-\beta)_r(\alpha)_r}{(\alpha+1)_r}\frac{t^r}{r!} = \frac{t^\alpha}{\alpha}\ {}_2F_1(1-\beta, \alpha; \alpha+1; t). \tag{1.7.6}$$

Hence the upper part,

$$B(\alpha,\beta;t) = \int_t^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} - b(\alpha,\beta;t). \tag{1.7.7}$$

Example 1.7.3. Obtain an integral representation for a ${}_2F_1$.

Solution 1.7.3. Consider the integral,

$$\int_0^1 x^{a-1}(1-x)^{c-a-1}(1-zx)^{-b}\mathrm{d}x, \text{ for } |z|<1$$
$$= \sum_{r=0}^{\infty}\frac{(z)^r}{r!}(b)_r\int_0^1 x^{a+r-1}(1-x)^{c-a-1}\mathrm{d}x, \text{(expanding } (1-zx)^{-b}$$

by binomial expansion)

$$= \sum_{r=0}^{\infty}\frac{(z)^r}{r!}(b)_r\frac{\Gamma(a+r)\Gamma(c-a)}{\Gamma(c+r)} \text{ (by using a type-1 beta integral)}$$
$$= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\sum_{r=0}^{\infty}\frac{(z)^r}{r!}\frac{(b)_r(a)_r}{(c)_r} \text{ (by writing } \Gamma(a+r)=(a)_r\Gamma(a))$$
$$= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}{}_2F_1(a,b;c;z).$$

That is,

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 x^{a-1}(1-x)^{c-a-1}(1-zx)^{-b}\mathrm{d}x \tag{1.7.8}$$
$$\text{for } \Re(a)>0,\ \Re(c-a)>0.$$

This is the famous integral representation for ${}_2F_1$. From the integral representation note that when $z=1$ one can evaluate the integral with the help of a type-1 beta integral. That is,

$$
\begin{aligned}
{}_2F_1\,(a, b; c; 1) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 x^{a-1}(1-x)^{c-a-b-1}\mathrm{d}x \\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \frac{\Gamma(a)\Gamma(c-a-b)}{\Gamma(c-b)}. \\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}
\end{aligned} \tag{1.7.9}
$$

when the arguments of all the gammas are positive. This is the famous summation formula for a ${}_2F_1$ series.

1.7.1. Evaluation of some contour integrals

Since the technique of Mellin and inverse Mellin transforms is frequently used for solving some problems in applied areas we will look into the evaluation of some contour integrals with the help of residue theorem. We will not go into the theory of analytic functions and residue calculus. We will need to know only how to apply the residue theorem for evaluating some integrands where the integrands contain gamma functions. In order to illustrate the technique let us redo a known result.

Example 1.7.4. Evaluate the contour integral, which is also an inverse Mellin transform,

$$
f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(a_1 - s)\cdots\Gamma(a_p - s)}{\Gamma(b_1 - s)\cdots\Gamma(b_q - s)} \Gamma(s)(-z)^{-s}\mathrm{d}s \tag{1.7.10}
$$

as the sum of the residues at the pole of $\Gamma(s)$.

Solution 1.7.4. The poles are at $s = -\nu, \nu = 0, 1, \cdots$. The residue at $s = -\nu$ is given by the following:

$$
\mathfrak{R}_\nu = \lim_{s\to-\nu} \left\{ (s+\nu)\Gamma(s) \frac{\Gamma(a_1 - s)\cdots\Gamma(a_p - s)}{\Gamma(b_1 - s)\cdots\Gamma(b_q - s)} (-z)^{-s} \right\}.
$$

By using the process in Example 1.3.2 we have,

$$\begin{aligned}\mathfrak{R}_\nu &= \frac{(-1)^\nu}{\nu!}\frac{\Gamma(a_1+\nu)\cdots\Gamma(a_p+\nu)}{\Gamma(b_1+\nu)\cdots\Gamma(b_q+\nu)}(-z)^\nu \\ &= \left\{\frac{\prod_{j=1}^p \Gamma(a_j)}{\prod_{j=1}^q \Gamma(b_j)}\right\}\frac{(a_1)_\nu\cdots(a_p)_\nu}{(b_1)_\nu\cdots(b_q)_\nu}\frac{z^\nu}{\nu!}.\end{aligned}$$

Hence the sum of the residues is given by,

$$\sum_{\nu=0}^{\infty}\mathfrak{R}_\nu = K\sum_{\upsilon=0}^{\infty}\frac{(a_1)_\nu\cdots(a_p)_\nu}{(b_1)_\nu\cdots(b_q)_\nu}\frac{z^\nu}{\nu!} = K\ {}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;z)$$

where K is the constant

$$K = \frac{\prod_{j=1}^p \Gamma(a_j)}{\prod_{j=1}^q \Gamma(b_j)}.$$

Thus $\frac{1}{K}$ times the right side in (1.7.10) is the Mellin-Barnes representation for a general hypergeometric function. If poles of higher orders are involved then one may use the general formula. If $\phi(z)$ has a pole of order m at $z = a$ then the residue at $z = a$, denoted by $\mathfrak{R}_a$, is given by the following formula:

$$\mathfrak{R}_a = \lim_{z\to a}\left\{\frac{1}{(m-1)!}[\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}(z-a)^m\phi(z)]\right\}. \tag{1.7.11}$$

Some illustrations of this formula will be given when we solve some problems in astrophysics later on.

Exercises 1.7.

1.7.1. For a Gauss' hypergeometric function ${}_2F_1$ derive the following relationships:

$$\begin{aligned}{}_2F_1(a,b;c;z) &= (1-z)^{-b}\ {}_2F_1(c-a,b;c;-\frac{z}{1-z}), z\neq 1 \\ &= (1-z)^{-a}\ {}_2F_1(a,c-b;c;-\frac{z}{1-z}), z\neq 1 \\ &= (1-z)^{c-a-b}\ {}_2F_1(c-a,c-b;c;z).\end{aligned}$$

1.7.2. Let x_1 and x_2 be independently distributed real scalar gamma random variables with the parameters $(\alpha_1, 1)$ and $(\alpha_2, 1)$ respectively. Let $u = x_1 x_2$. Evaluate the density of u by using Mellin transformation technique when α_1 and α_2 do not differ by integers or zero.

1.7.3. Let x_1 and x_2 be independently distributed real type-1 beta random variables with the parameters (α_1, β_1) and (α_2, β_2) respectively. Let $u = x_1 x_2$. Evaluate the density of u by using Mellin transform technique if α_1 and α_2 do not differ by integers or zero.

1.7.4. Repeat the problem in Exercise 1.7.3 if x_1 and x_2 are type-2 beta distributed, where $\alpha_1 - \alpha_2 \neq \pm\lambda, \lambda = 0, 1, \cdots \beta_1 - \beta_2 \neq \pm\nu, \nu = 0, 1, 2, \cdots$.

1.7.5. Let $f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(\alpha - s)\Gamma(s)x^{-s}\mathrm{d}s$. Evaluate $f(x)$ as the sum of residues at the poles of $\Gamma(s)$. Then evaluate it again at the poles of $\Gamma(\alpha - s)$. Then compare the two results. In the first case we get the function for $|x| < 1$ and in the case for $|x| > 1$.

1.8. Meijer's G-function

A generalization of the hypergeometric function in the real scalar case is Meijer's G-Function. It is defined in terms of a Mellin-Barnes integral.

Notation 1.8.1.

$$G_{p,q}^{m,n}[z|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}] = G_{p,q}^{m,n}[z|_{(b_q)}^{(a_p)}] = G_{p,q}^{m,n}(z) = G(z).$$

Definition 1.8.1. G-function.

$$G_{p,q}^{m,n}[z|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}] = \frac{1}{2\pi i}\int_L \frac{\left\{\prod_{j=1}^m \Gamma(b_j + s)\right\}\left\{\prod_{j=1}^n \Gamma(1 - a_j - s)\right\} z^{-s}\mathrm{d}s}{\left\{\prod_{j=m+1}^q \Gamma(1 - b_j - s)\right\}\left\{\prod_{j=n+1}^p \Gamma(a_j + s)\right\}} \tag{1.8.1}$$

where L is a contour separating the poles of $\Gamma(b_j + s)$, $j = 1, \cdots, m$ from those of $\Gamma(1 - a_j - s)$, $j = 1, \cdots, n$. Three types of contours are described and the conditions of existence for the G-function are discussed in Mathai (1993). The simplified conditions are the following: $G(z)$ exists for the following situations:

$$\begin{aligned} &\text{(i)} \quad q \geq 1, q > p, \ \text{for all} \ \ z, z \neq 0 \\ &\text{(ii)} \quad q \geq 1, q = p, \ \text{for } |z| < 1 \\ &\text{(iii)} \quad p \geq 1, p > q, \ \text{for all} \ \ z, z \neq 0 \\ &\text{(iv)} \quad p \geq 1, p = q, \ \text{for } |z| > 1. \end{aligned} \tag{1.8.2}$$

Example 1.8.1. Evaluate

$$f(x) = G_{1,1}^{1,0}\left[x\Big|_{\alpha}^{\alpha+\beta+1}\right].$$

Solution 1.8.1. As per our notation, $m = 1, n = 0, p = 1, q = 1.$

$$G_{1,1}^{1,0}\left[x\Big|_{\alpha}^{\alpha+\beta+1}\right] = \frac{1}{2\pi i}\int_L \frac{\Gamma(\alpha+s)}{\Gamma(\alpha+\beta+1+s)}x^{-s}\mathrm{d}s.$$

As per situation (ii) above we should obtain a convergent function for $|x| < 1$ if we evaluate the integral as the sum of the residues at the poles of $\Gamma(\alpha+s)$. The poles are at $s = -\alpha - \nu, \nu = 0, 1, \cdots$ and the sum of the residues

$$\sum_{\nu=0}^{\infty}\mathfrak{R}_\nu = \sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}\frac{x^{\nu+\alpha}}{\Gamma(\beta+1-\nu)}; \Gamma(\beta+1-\nu) = \frac{(-1)^\nu\Gamma(\beta+1)}{(-\beta)_\nu}.$$

$$G_{1,1}^{1,0}[x|_{\alpha}^{\alpha+\beta+1}] = \frac{x^\alpha}{\Gamma(\beta+1)}\sum_{\nu=0}^{\infty}\frac{(-\beta)_\nu x^\nu}{\nu!} = \frac{x^\alpha}{\Gamma(\beta+1)}(1-x)^\beta, |x| < 1 \tag{1.8.3}$$

for $\mathfrak{R}(\beta+1) > 0.$

Example 1.8.2. Let $u = x_1x_2\cdots x_p$ where $x_1, \cdots, x_p$ are independently distributed real random variables with (1) : x_j gamma distributed with parameters $(\alpha_j, 1), j = 1, \cdots, p$; (2) : x_j type-1 beta distributed with parameters $(\alpha_j, \beta_j), j = 1, \cdots, p$; (3) : x_j is type-2 beta distributed with parameters $(\alpha_j, \beta_j), j = 1, \cdots, p$. Evaluate the density of u in (1),(2) and (3).

Solution 1.8.2. Taking the $(s-1)^{th}$ moment of u or the Mellin transform of the density of u we have the following:

$$E(u^{s-1}) = E(x_1 \cdots x_p)^{s-1} = E(x_1^{s-1} \cdots x_p^{s-1}) = E(x_1^{s-1}) \cdots E(x_p^{s-1})$$

due to independence

$$= \prod_{j=1}^{p} E(x_j^{s-1}) = \prod_{j=1}^{p} \frac{\Gamma(\alpha_j + s - 1)}{\Gamma(\alpha_j)}, \Re(\alpha_j + s - 1) > 0, j = 1, \cdots, p$$

for case (1)

$$= \prod_{j=1}^{p} \frac{\Gamma(\alpha_j + s - 1)}{\Gamma(\alpha_j)} \frac{\Gamma(\alpha_j + \beta_j)}{\Gamma(\alpha_j + \beta_j + s - 1)}, \Re(\alpha_j + s - 1) > 0, j = 1, \cdots, p$$

for case (2)

$$= \prod_{j=1}^{p} \frac{\Gamma(\alpha_j + s - 1)}{\Gamma(\alpha_j)} \frac{\Gamma(\beta_j - s + 1)}{\Gamma(\beta_j)}, \Re(\alpha_j + s - 1) > 0, \Re(\beta_j - s + 1) > 0,$$

$j = 1, \cdots, p$ for case (3).

Let the densities be denoted by $g_1(u), g_2(u)$ and $g_3(u)$ respectively. They are available from the respective inverse Mellin transforms which can be written as G-functions as follows:

$$\begin{aligned} g_1(u) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left\{ \prod_{j=1}^{p} \frac{\Gamma(\alpha_j - 1 + s)}{\Gamma(\alpha_j)} \right\} u^{-s} \mathrm{d}s \\ &= \frac{1}{\left\{ \prod_{j=1}^{p} \Gamma(\alpha_j) \right\}} G_{0,p}^{p,0}\left[u\big|_{\alpha_j - 1, j=1,\cdots,p}\right], \text{ for } u > 0, \Re(\alpha_j) > 0, j = 1, \cdots, p \end{aligned} \tag{1.8.4}$$

and zero elsewhere.

$$
\begin{aligned}
g_2(u) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{\prod_{j=1}^{p}\frac{\Gamma(\alpha_j+s-1)}{\Gamma(\alpha_j)}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j+\beta_j+s-1)}\right\}u^{-s}\mathrm{d}s\\
&= \left\{\prod_{j=1}^{p}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)}\right\}G_{p,p}^{p,0}[u\big|_{\alpha_j-1,j=1,\cdots,p}^{\alpha_j+\beta_j-1,j=1,\cdots,p}],\ 0<u<1, \qquad (1.8.5)\\
&\quad \Re(\alpha_j)>0, \Re(\beta_j)>0, \text{and zero elsewhere .}
\end{aligned}
$$

$$
\begin{aligned}
g_3(u) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{\prod_{j=1}^{p}\frac{\Gamma(\alpha_j+s-1)}{\Gamma(\alpha_j)}\frac{\Gamma(\beta_j-s+1)}{\Gamma(\beta_j)}\right\}u^{-s}\mathrm{d}s \qquad (1.8.6)\\
&= \frac{1}{\left\{\prod_{j=1}^{p}\Gamma(\alpha_j)\Gamma(\beta_j)\right\}}G_{p,p}^{p,p}[u\big|_{\alpha_j-1,j=1,\cdots,p}^{-\beta_j,j=1,\cdots,p}],\ u>0,\\
&\quad \Re(\alpha_j)>0, \Re(\beta_j)>0,\ j=1,\cdots,p \text{ and zero elsewhere.}
\end{aligned}
$$

Example 1.8.3. Evaluate the following integral, a particular case of which is the reaction rate integral in astrophysics.

$$
I(\alpha,a,b,\rho)=\int_0^\infty x^{\alpha-1}\mathrm{e}^{-ax-bx^{-\rho}}\mathrm{d}x,\ a>0, b>0, \rho>0. \qquad (1.8.7)
$$

Solution 1.8.3. Since the integrand can be taken as a product of positive integrable functions we can apply statistical distribution theory to evaluate this integral or such similar integrals. The procedure to be discussed here is suitable for a wide variety of problems. Let x_1 and x_2 be two real scalar random variables with density functions $f_1(x_1)$ and $f_2(x_2)$. Let $u = x_1x_2$ and let x_1 and x_2 be independently distributed. Then the joint density of x_1 and x_2, denoted by $f(x_1, x_2)$, is the product of the marginal densities due to statistical independence of x_1 and x_2. That is,

$$
f(x_1,x_2) = f_1(x_1)f_2(x_2).
$$

Consider the transformation $u = x_1x_2$ and $v = x_1 \Rightarrow \mathrm{d}x_1\wedge\mathrm{d}x_2 = \frac{1}{v}\mathrm{d}u\wedge\mathrm{d}v$. Hence the joint density of u and v, denoted by $g(u, v)$, is available as,

$$g(u,v) = \frac{1}{v} f_1(v) f_2(\frac{u}{v}). \tag{1.8.8}$$

Then the density of u denoted by $g_1(u)$, is available by integrating out v from $g(u,v)$. That is,

$$g_1(u) = \int_v \frac{1}{v} f_1(v) f_2(\frac{u}{v}) \mathrm{d}v. \tag{1.8.9}$$

Here (1.8.8) and (1.8.9) are general results and the method described here is called the *method of transformation of variables* for obtaining the density of $u = x_1 x_2$. Now, consider (1.8.7). Let

$$f_1(x_1) = c_1 x_1^{\alpha} \mathrm{e}^{-a x_1} \text{ and } f_2(x_2) = c_2 \mathrm{e}^{-z x_2^{\rho}}, 0 \le x_1 < \infty, 0 \le x_2 < \infty \tag{1.8.10}$$

$a > 0, z > 0$, where c_1 and c_2 are the normalizing constants. These normalizing constants can be evaluated by using the property.

$$1 = \int_0^{\infty} f_1(x_1) \mathrm{d}x_1 \text{ and } 1 = \int_0^{\infty} f_2(x_2) \mathrm{d}x_2.$$

Since we do not need the explicit forms of c_1 and c_2 we will not evaluate them here. With the f_1 and f_2 in (1.8.10) let us evaluate (1.8.9). We have

$$g_1(u) = c_1 c_2 \int_{v=0}^{\infty} \frac{1}{v} v^{\alpha} \mathrm{e}^{-av} \mathrm{e}^{-z(\frac{u}{v})^{\rho}} \mathrm{d}v = c_1 c_2 \int_{v=0}^{\infty} v^{\alpha-1} \mathrm{e}^{-av} \mathrm{e}^{-(zu^{\rho})v^{-\rho}} \mathrm{d}v. \tag{1.8.11}$$

Note that with $b = zu^{\rho}$, (1.8.11) is (1.8.7) multiplied by c_1 and c_2. Thus, we have identified the integral to be evaluated as a constant multiple of the density of u. This density of u is unique. Let us evaluate the density through Mellin and inverse Mellin transform technique.

$$E(u^{s-1}) = E(x_1^{s-1}) E(x_2^{s-1})$$

due to statistical independence of x_1 and x_2. But

$$E(x_1^{s-1}) = c_1 \int_0^{\infty} x_1^{\alpha+s-1} \mathrm{e}^{-a x_1} \mathrm{d}x_1 = c_1 a^{-(\alpha+s)} \Gamma(\alpha + s), \Re(\alpha + s) > 0 \tag{1.8.12}$$

and

$$E(x_2^{s-1}) = c_2 \int_0^{\infty} x_2^{s-1} \mathrm{e}^{-z x_2^{\rho}} \mathrm{d}x_2 = \frac{c_2}{\rho z^{s/\rho}} \int_0^{\infty} y^{\frac{s}{\rho}-1} \mathrm{e}^{-y} \mathrm{d}y = \frac{c_2}{\rho z^{s/\rho}} \Gamma(\frac{s}{\rho}), \Re(s) > 0. \tag{1.8.13}$$

Hence

$$E(u^{s-1}) = c_1c_2\frac{a^{-\alpha}}{\rho}(az^{\frac{1}{\rho}})^{-s}\Gamma(\alpha+s)\Gamma(\frac{s}{\rho}), \tag{1.8.14}$$

Therefore, the density of u, denoted by $g_1(u)$, is available from the inverse Mellin transform.

$$g_1(u) = c_1c_2\frac{a^{-\alpha}}{\rho}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\alpha+s)\Gamma(\frac{s}{\rho})(az^{\frac{1}{\rho}}u)^{-s}\,\mathrm{d}s. \tag{1.8.15}$$

Now, compare (1.8.15) with (1.8.11) to obtain the following:

$$\int_0^\infty v^{\alpha-1}\mathrm{e}^{-av}\mathrm{e}^{-(zu^\rho)v^{-\rho}}\mathrm{d}v = \frac{a^{-\alpha}}{\rho}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\alpha+s)\Gamma(\frac{s}{\rho})(az^{\frac{1}{\rho}}u)^{-s}\,\mathrm{d}s. \tag{1.8.16}$$

On the right side in (1.8.15) the coefficient of s in $\Gamma(\frac{s}{\rho})$ is $\frac{1}{\rho}\neq 1$. Hence (1.8.15) is not a G-function but it can be written as an H-function, which will be considered next. In reaction rate theory in physics $\rho=\frac{1}{2}$ and then

$$\Gamma(\frac{s}{\rho}) = \Gamma(2s) = \pi^{\frac{1}{2}}2^{2s-1}\Gamma(s)\Gamma\Big(s+\frac{1}{2}\Big)$$

by using the duplication formula for gamma functions. Then the right side of (1.8.16) reduces to

$$\frac{1}{2\rho a^\alpha\pi^{\frac{1}{2}}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\alpha+s)\Gamma(s)\Gamma\Big(s+\frac{1}{2}\Big)\Big(\frac{auz^{1/\rho}}{4}\Big)^{-\rho}\mathrm{d}s$$

$$= \frac{1}{2\rho a^\alpha\pi^{\frac{1}{2}}}G_{0,3}^{3,0}\Big[\frac{auz^{1/\rho}}{4}\Big|_{\alpha,0,\frac{1}{2}}\Big], u>0.$$

But

$$b = zu^\rho \Rightarrow \frac{auz^{1/\rho}}{4} = \frac{ab^{1/\rho}}{4}.$$

Hence, for $\rho=\frac{1}{2}$,

$$\int_0^\infty v^{\alpha-1}\mathrm{e}^{-av-bv^{-\rho}}\mathrm{d}v = \frac{1}{2\rho a^\alpha\pi^{\frac{1}{2}}}G_{0,3}^{3,0}\Big[\frac{ab^{1/\rho}}{4}\Big|_{\alpha,0,\frac{1}{2}}\Big] \text{ for } \rho=\frac{1}{2}$$

$$= \frac{1}{a^\alpha\pi^{\frac{1}{2}}}G_{0,3}^{3,0}\Big[\frac{ab^2}{4}\Big|_{\alpha,0,\frac{1}{2}}\Big], u>0. \tag{1.8.17}$$

Exercises 1.8.

Write down the Mellin-Barnes representations in Exercises 1.8.1.- 1.8.5 where the series forms are given. Here is an illustration.

$$
\begin{aligned}
{}_1F_0(\alpha;\,;x) &= \sum_{r=0}^{\infty}(\alpha)_r\frac{x^r}{r!} = \frac{1}{\Gamma(\alpha)}\sum_{r=0}^{\infty}\Gamma(\alpha+r)\frac{x^r}{r!} \\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\alpha-s)\Gamma(s)(-x^{-s})\mathrm{d}s.
\end{aligned}
$$

The last expression is the Mellin-Barnes representation for the series form ${}_1F_0(\alpha;\,;x)$.

1.8.1. ${}_0F_0(;\,;-z) = \mathrm{e}^{-z} = \sum_{r=0}^{\infty}\frac{(-z)^r}{r!}$ (Exponential Series)

1.8.2. ${}_2F_1(a,b;c;z) = \sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r}\frac{(z)^r}{r!}$ (Gauss' hypergeometric series)

1.8.3. ${}_1F_1(a;b;z) = \sum_{r=0}^{\infty}\frac{(a)_r}{(b)_r}\frac{(z)^r}{r!}$ (Confluent hypergeometric series)

1.8.4. $\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{(z/2)^{\nu+2r}}{\Gamma(\nu+r+1)}$ (Bessel function $J_\nu(z)$)

1.8.5. $\sum_{r=0}^{\infty}\frac{(z/2)^{\nu+2r}}{r!\Gamma(\nu+r+1)}$ (Bessel function $I_\nu(z)$).

Write the series forms from the Mellin-Barnes representation in Exercise 1.8.6 and list the conditions for convergence and existence also.

1.8.6. $\frac{\Gamma(1+2\nu)}{\Gamma(\frac{1}{2}+\nu-\mu)}\mathrm{e}^{-z/2}z^{\nu+\frac{1}{2}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\Gamma(\frac{1}{2}+\nu-\mu-s)}{\Gamma(1+2\nu-s)}(-z)^{-s}\mathrm{d}s$ (Whittaker function $M_{\mu,\nu}(z)$)

Represent the following in Exercises 1.8.7 to 1.8.10 as G-functions and write down the conditions.

1.8.7. $z^{\beta}(1+az^{\alpha})^{-1}$

1.8.8. $z^{\beta}(1+az^{\alpha})^{-\gamma}$

1.8.9. $(a)\ \sin z$; $(b)\ \cos z$; $(c)\ \sinh z$; $(d)\ \cosh z$

1.8.10. $(a)\ \ln(1\pm z)$; $(b)\ln\left(\frac{1+z}{1-z}\right)$.

1.9. The H-function

This function is a generalization of the G-function. This was available in the literature as a Mellin -Barnes integral but Charles Fox made a detailed study of it in 1960's and hence the function is called Fox's H-function. The Mellin-Barnes representation is the following:

Notation 1.9.1. H-function

$$H_{p,q}^{m,n}\left[z\Big|_{(b_1,\beta_1),\cdots,(b_q,\beta_q)}^{(a_1,\alpha_1),\cdots,(a_p,\alpha_p)}\right] = H_{p,q}^{m,n}\left[z\Big|_{[(b_q,\beta_q)]_q}^{[(a_p,\alpha_p)]}\right] = H_{p,q}^{m,n}(z) = H(z).$$

Definition 1.9.1.

$$H_{p,q}^{m,n}\left[z\Big|_{(b_1,\beta_1),\cdots,(b_q,\beta_q)}^{(a_1,\alpha_1),\cdots,(a_p,\alpha_p)}\right] = \frac{1}{2\pi i}\int_L \frac{\left\{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\right\}\left\{\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)\right\}}{\left\{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\right\}\left\{\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)\right\}} z^{-s}\mathrm{d}s \tag{1.9.1}$$

where $\alpha_1,\cdots,\alpha_p,\beta_1,\cdots,\beta_q$ are real positive numbers (integers, rationals or irrationals), a_j's and b_j's are, in general, complex quantities, $i=\sqrt{-1}$ and the contour L separates the poles of $\Gamma(b_j+\beta_j s)$, $j=1,\cdots,m$ from those of $\Gamma(1-a_j-\alpha_j s)$, $j=1,\cdots,n$. Three paths L, similar to the ones for a G-function, can be given for the H-function also. Details of the existence conditions, various properties and applications may be seen from Mathai and Saxena (1978) and Mathai (1993). A simplified set of existence conditions is the following: Let,

$$\mu = \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j \text{ and } \beta = \left\{\prod_{j=1}^{q}\alpha_j^{\alpha_j}\right\}\left\{\prod_{j=1}^{q}\beta_j^{-\beta_j}\right\}. \tag{1.9.2}$$

The H-function exists for the following cases:

$$\begin{aligned} &\text{(i)} \quad q \geq 1, \mu > 0, \text{ for all } z, z \neq 0 \\ &\text{(ii)} \quad q \geq 1, \mu = 0, \text{ for } |z| < \beta^{-1} \\ &\text{(iii)} \quad p \geq 1, \mu < 0, \text{ for all } z, z \neq 0 \\ &\text{(iv)} \quad p \geq 1, \mu = 0, \text{ for } |z|, z > \beta^{-1}. \end{aligned} \tag{1.9.3}$$

Two special cases, which follow from the definition itself, may be noted. When $\alpha_1 = 1 = \cdots = \alpha_p = \beta_1 = 1 = \cdots = \beta_q$ then the H-function reduces to a G-function. When all the α_j's and β_j's are rational numbers, that is ratios of two positive integers since by definition the α_j's and β_j's are positive real numbers, we may make a transformation $\frac{s}{u} = s_1$ where u is the common denominator for all the $\alpha_j, j = 1, \cdots, p$ and $\beta_j, j = 1, \cdots, q$. Under this transformation each coefficient of s_1 in each gamma in (1.9.1) becomes a positive integer. Then we may expand all the gammas by using the multiplication formula for gamma functions. Then the coefficients of s in every gamma becomes ± 1 and then the H-function becomes a G-function. An illustration of this aspect was seen in Example 1.8.3.

Example 1.9.1. Evaluate the following reaction rate integral in physics and write it as an H-function.

$$I(\alpha, a, b, \rho) = \int_0^{\infty} x^{\alpha-1} e^{-ax-bx^{-\rho}} dx.$$

Solution 1.9.1. From (1.8.16) in Example 1.8.3 we have

$$\int_0^{\infty} x^{\alpha-1} e^{-ax-bx^{-\rho}} dx = \frac{a^{-\alpha}}{\rho} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\alpha + s)\Gamma(\frac{s}{\rho})(ab^{1/\rho})^{-s} ds. \tag{1.9.4}$$

Writing the right side with the help of (1.9.1) we have the following:

$$\int_0^{\infty} x^{\alpha-1} e^{-ax-bx^{-\rho}} dx = \frac{1}{\rho a^{\alpha}} H_{0,2}^{2,0}\left[ab^{\frac{1}{\rho}}\Big|_{(\alpha,1),(0,\frac{1}{\rho})}\right]. \tag{1.9.5}$$

Example 1.9.2. Let $x_1, \cdots, x_k$ be independently distributed real scalar gamma random variables with the parameters $(\alpha_j, 1), j = 1, \cdots, k$. Let $\gamma_1, \cdots, \gamma_k$ be real constants. Let

$$u = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_k^{\gamma_k}.$$

Evaluate the density of u.

Solution 1.9.2. Let us take the $(s-1)^{th}$ moment of u or the Mellin transform of the density of u.

$$E(u^{s-1}) = E[x_1^{\gamma_1} \cdots x_k^{\gamma_k}]^{s-1} = E(x_1^{\gamma_1(s-1)}) \cdots E[x_k^{\gamma_k(s-1)}]$$

due to independence. But for a real gamma random variable, with parameters $(\alpha_j, 1)$, the $[\gamma_j(s-1)]^{th}$ moment is the following:

$$E[x_j^{\gamma_j(s-1)}] = \frac{\Gamma(\alpha_j + \gamma_j(s-1))}{\Gamma(\alpha_j)} = \frac{\Gamma(\alpha_j - \gamma_j + \gamma_j s)}{\Gamma(\alpha_j)} \text{ for } \Re(\alpha_j + \gamma_j(s-1)) > 0. \tag{1.9.6}$$

Then

$$E(u^{s-1}) = \prod_{j=1}^{k} \frac{\Gamma(\alpha_j - \gamma_j + \gamma_j s)}{\Gamma(\alpha_j)}.$$

The density of u, denoted by $g(u)$, is available from the inverse Mellin transform. That is,

$$g(u) = \frac{1}{\left\{\prod_{j=1}^{k} \Gamma(\alpha_j)\right\}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left\{\prod_{j=1}^{k} \Gamma(\alpha_j - \gamma_j + \gamma_j s)\right\} u^{-s} \mathrm{d}s$$

$$= \begin{cases} \frac{1}{\left\{\prod_{j=1}^{k} \Gamma(\alpha_j)\right\}} H_{0,k}^{k,0}[u|_{(\alpha_j - \gamma_j, \gamma_j), j=1,\cdots,k}], u > 0, \\ 0, \quad \text{elsewhere.} \end{cases}$$

This is the density function for the product of arbitrary powers of independently distributed real scalar gamma random variables.

By using similar procedures one can obtain and write in terms of H-functions, products of arbitrary powers of real scalar type-1 beta and type-2 beta random variables or arbitrary powers of products and ratios of real scalar gamma, type-1, type-2 beta or other such positive variables. Some details may by seen from Mathai (1993) and Mathai and Saxena (1978).

Exercises 1.9.

1.9.1. Prove that

$$H_{p,q}^{m,n}[z|_{(b_1,\beta_1),\cdots,(b_q,\beta_q)}^{(a_1,\alpha_1),\cdots,(a_p,\alpha_p)}] = H_{q,p}^{n,m}[\frac{1}{z}|_{(1-a_1,\alpha_1),\cdots,(1-a_p,\alpha_p)}^{(1-b_1,\beta_1),\cdots,(1-b_q,\beta_q)}].$$

1.9.2. Prove that

$$z^{\sigma}H_{p,q}^{m,n}[z|_{(b_1,\beta_1),\cdots,(b_q,\beta_q)}^{(a_1,\alpha_1),\cdots,(a_p,\alpha_p)}] = H_{p,q}^{m,n}[z|_{(b_1+\sigma\beta_1,\beta_1),\cdots,(b_q+\sigma\beta_q,\beta_q)}^{(a_1+\sigma\alpha_1,\alpha_1),\cdots,(a_p+\sigma\alpha_p,\alpha_p)}].$$

1.9.3. Evaluate the Mellin-Barnes integral

$$E_{\alpha}(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)}{\Gamma(1-\alpha s)}(-z)^{-s}\mathrm{d}s \tag{1.9.7}$$

and show that $E_{\alpha}(z)$ is the Mittag-Leffler series

$$E_{\alpha}(z) = \sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\alpha r+1)}. \tag{1.9.8}$$

1.9.4. Evaluate the Laplace transform of $E_{\alpha}(z^{\alpha})$ of Exercise 1.9.3, in (1.9.8), with parameter p.

1.9.5. A generalization of Mittag-Leffler function $E_{\alpha,\beta}(z) = \sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\alpha r+\beta)}$. Evaluate the Laplace transform of $t^{\beta-1}E_{\alpha,\beta}(z^{\alpha})$.

1.9.6. If $\alpha = m, m = 1, 2, \cdots$ in (1.9.8) show that

$$E_{\alpha}(z) = (2\pi)^{\frac{m-1}{2}}m^{-\frac{1}{2}}{}_0F_{m-1}(;\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m};\frac{z}{m^m})\frac{1}{\Gamma(\frac{1}{m}),\Gamma(\frac{2}{m}),\cdots\Gamma(\frac{m-1}{m})}.$$

1.9.7. Write $E_{\alpha}(z)$ as an H-function.

1.9.8. If $\alpha = m, m = 1, 2, \cdots$ write down $E_{\alpha}(z)$ as a G-function.

1.9.9. Let x_1 and x_2 be independently distributed real gamma random variables with the parameters $(\alpha, 1), (\alpha + \frac{1}{2}, 1)$ respectively. Let $u = x_1 x_2$. Evaluate the density of u and show that the density of u, denoted by $g(u)$, is given by the following:

$$g(u) = \frac{2^{2\alpha-1}}{\Gamma(2\alpha)} u^{\alpha-1} e^{-2u^{\frac{1}{2}}}, u \geq 0 \quad \text{and zero elsewhere.}$$

1.9.10. Let x_1, x_2, x_3 be independently distributed real gamma random variables with the parameters $(\alpha, 1), (\alpha + \frac{1}{3}, 1), (\alpha + \frac{2}{3}, 1)$ respectively. Let $u = x_1 x_2 x_3$. Evaluate the density of u and show that it can be written as an H-function of the following type, where $g(u)$ denotes the density of u.

$$g(u) = \frac{27}{\Gamma(3\alpha)} H_{0,1}^{1,0}[27u\big|_{(3\alpha-3,3)}], u \geq 0 \text{ and zero elsewhere.}$$

1.10. Lauricella Functions and Appell's Functions

Another set of multivariable functions in frequent use in applied areas is the set of Lauricalla functions, and special cases of those are the Appell's functions. Lauricella functions f_A, f_B, f_C, and f_D are the following:

Definition 1.10.1. **Lauricella function** f_A

$$f_A(a, b_1, \cdots, b_n; c_1, \cdots, c_n; x_1, \cdots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!} \tag{1.10.1}$$

for $|x_1| + \cdots + |x_n| < 1$.

Definition 1.10.2. **Lauricella function** f_B

$$f_B(a_1, \cdots, a_n, b_1, \cdots, b_n; c; x_1, \cdots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a_1)_{m_1} \cdots (a_n)_{m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!} \tag{1.10.2}$$

for $|x_1| < 1, |x_2| < 1, \cdots, |x_n| < 1$.

Definition 1.10.3. **Lauricella function** f_C

$$f_C(a, b; c_1, \cdots, c_n; x_1, \cdots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b)_{m_1+\cdots+m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}} \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!} \tag{1.10.3}$$

for $|\sqrt{x_1}| + \cdots + |\sqrt{x_n}| < 1$.

Definition 1.10.4. **Lauricella function** f_D

$$f_D(a, b_1, \cdots, b_n; c; x_1, \cdots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!} \tag{1.10.4}$$

for $|x_1| < 1, |x_2| < 1, \cdots, |x_n| < 1$.

When $n = 2$ we have Appell's functions F_1, F_2, F_3, F_4. Also when $n = 1$ all these functions reduce to a Gauss' hypergeometric function ${}_2F_1$. We will list some of the basic properties of Lauricella functions.

1.10.1. Some properties of f_A

$$\begin{aligned}\int_0^1 \cdots \int_0^1 & u_1^{b_1-1}\cdots u_n^{b_n-1}(1-u_1)^{c_1-b_1-1}\cdots(1-u_n)^{c_n-b_n-1} \\ & \times (1 - u_1x_1 - \cdots - u_nx_n)^{-a}\, \mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n \\ & = \left\{\prod_{j=1}^{n} \frac{\Gamma(b_j)\Gamma(c_j - b_j)}{\Gamma(c_j)}\right\} f_A(a, b_1, \cdots, b_n; c_1, \cdots, c_n; x_1, \cdots, x_n),\end{aligned} \tag{1.10.5}$$

for $\Re(b_j) > 0$, $\Re(c_j - b_j) > 0$, $j = 1, \cdots, n$.

The result can be easily established by expanding the factor $(1 - u_1x_1 - \cdots - u_nx_n)^{-a}$ by using a multinomial expansion and then integrating out u_j, $j = 1, \cdots, n$ with the help of type-1 beta integrals.

$$\int_0^\infty e^{-t} t^{a-1} {}_1F_1(b_1; c_1; x_1 t) {}_1F_1(b_2; c_2; x_2 t) \cdots {}_1F_1(b_n; c_n; x_n t) \mathrm{d}t \tag{1.10.6}$$
$$= \Gamma(a) f_A(a, b_1, \cdots, b_n; c_1, \cdots, c_n; x_1, \cdots, x_n), \text{ for } \Re(a) > 0.$$

This can be established by taking the series forms for ${}_1F_1$'s and then integrating out t.

$$\frac{1}{(2\pi i)^n} \int \cdots \int \frac{\Gamma(a + t_1 + \cdots + t_n)\Gamma(b_1 + t_1) \cdots \Gamma(b_n + t_n)}{\Gamma(c_1 + t_1) \cdots \Gamma(c_n + t_n)}$$
$$\times \Gamma(-t_1) \cdots \Gamma(-t_n)(-x_1)^{t_1} \cdots (-x_n)^{t_n} \, \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n \tag{1.10.7}$$
$$= \Gamma(a) \frac{\Gamma(b_1) \cdots \Gamma(b_n)}{\Gamma(c_1) \cdots \Gamma(c_n)} \, f_A(a, b_1, \cdots, b_n; c_1, \cdots, c_n; x_1, \cdots, x_n), i = \sqrt{-1}.$$

This can be established by evaluating the integrand as the sum of the residues at the poles of $\Gamma(-t_1), \cdots, \Gamma(-t_n)$, one by one.

1.10.2. Some properties of f_B

$$\int \cdots \int t_1^{a_1-1} \cdots t_n^{a_n-1} (1 - t_1 - \cdots - t_n)^{c-a_1-\cdots-a_n-1} \tag{1.10.8}$$
$$\times (1 - t_1 x_1)^{-b_1} \cdots (1 - t_n x_n)^{-b_n} \, \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n$$
$$= \frac{\Gamma(a_1) \cdots \Gamma(a_n)\Gamma(c - a_1 - \cdots - a_n)}{\Gamma(c)} f_B(a_1, \cdots, a_n, b_1, \cdots, b_n; c; x_1, \cdots, x_n),$$

for $\Re(a_j) > 0, j = 1, \cdots, n, \Re(c - a_1 - \cdots - a_n) > 0, t_j > 0, j = 1, \cdots, n,$ and $1 - t_1 - \cdots - t_n > 0$.

This result can be established by opening up $(1 - t_j x_j)^{-b_j}, j = 1, \cdots, n$ by using binomial expansions and then integrating out $t_1, \cdots, t_n$ with the help of a type-1 Dirichlet integral of Section 1.8.

$$\int_0^\infty \cdots \int_0^\infty s_1^{a_1-1} \cdots s_n^{a_n-1} t_1^{b_1-1} \cdots t_n^{b_n-1} e^{-s_1-\cdots-s_n-t_1-\cdots-t_n} \tag{1.10.9}$$
$$\times {}_0F_1 (; c; s_1 t_1 x_1 + \cdots + s_n t_n x_n) \, \mathrm{d}s_1 \wedge \cdots \wedge \mathrm{d}s_n \wedge \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n$$
$$= \left\{ \prod_{j=1}^n \Gamma(a_j)\Gamma(b_j) \right\} f_B(a_1 \cdots, a_n, b_1, \cdots, b_n; c; x_1, \cdots, x_n),$$

for $\Re(a_j) > 0, \quad \Re(b_j) > 0, j = 1, \cdots, n.$

First, open up the ${}_0F_1$ as a power series in $(s_1t_1x_1+\cdots+s_nt_nx_n)^k$. Since k is a positive integer open up by using a multinomial expansion. Then integrate out $s_1,\cdots,s_k$ and $t_1,\cdots,t_k$ by using gamma functions, to see the result.

$$\begin{aligned}
&\frac{1}{(2\pi i)^n}\int\cdots\int\frac{\Gamma(a_1+t_1)\cdots\Gamma(a_n+t_n)\Gamma(b_1+t_1)\cdots\Gamma(b_n+t_n)}{\Gamma(c+t_1+\cdots+t_n)} \qquad (1.10.10)\\
&\qquad\times\Gamma(-t_1)\cdots\Gamma(-t_n)(-x_1)^{t_1}\cdots(-x_n)^{t_n}\,\mathrm{d}t_1\wedge\cdots\wedge\mathrm{d}t_n,\ i=\sqrt{-1}\\
&\qquad=\left\{\prod_{j=1}^{n}\frac{\Gamma(a_j)\Gamma(b_j)}{\Gamma(c)}\right\}f_B(a_1,\cdots,a_n,b_1,\cdots,b_n;c;x_1,\cdots,x_n).
\end{aligned}$$

Assume that $a_j-b_j\neq\pm\nu,\nu=0,1,\cdots$. Then evaluate the integrand as the sum of the residues at the poles of $\Gamma(-t_1),\cdots,\Gamma(-t_n)$, one by one, to obtain the result.

1.10.3. Some properties of f_C

$$\begin{aligned}
&\int_0^\infty\cdots\int_0^\infty s^{a-1}t^{b-1}\mathrm{e}^{-s-t}\ {}_0F_1\ (;c_1;x_1st)\cdots\ {}_0F_1\ (;c_n;x_nst)\mathrm{d}s\wedge\mathrm{d}t \qquad (1.10.11)\\
&\quad=\Gamma(a)\Gamma(b)f_C(a,b;c_1,\cdots,c_n;x_1,\cdots,x_n),\quad\text{for }\ \Re(a)>0,\Re(b)>0.
\end{aligned}$$

Open up the ${}_0F_1$'s, then integrate out t and s with the help of gamma integrals to see the result.

$$\begin{aligned}
&\frac{1}{(2\pi i)^n}\int\cdots\int\frac{\Gamma(a+t_1+\cdots+t_n)\Gamma(b+t_1+\cdots+t_n)}{\Gamma(c_1+t_1)\cdots\Gamma(c_n+t_n)}\\
&\qquad\times\ \Gamma(-t_1)\cdots\Gamma(-t_n)(-x_1)^{t_1}\cdots(-x_n)^{t_n}\mathrm{d}t_1,\wedge\cdots\wedge\mathrm{d}t_n\\
&\qquad=\frac{\Gamma(a)\Gamma(b)}{\Gamma(c_1)\cdots\Gamma(c_n)}f_C(a,b;c_1,\cdots,c_n;x_1,\cdots,x_n),i=\sqrt{-1}. \qquad (1.10.12)
\end{aligned}$$

Evaluate the integrand as the sum of the residues at the poles of $\Gamma(-t_1),\cdots,\Gamma(-t_n)$, one by one, to obtain the result.

1.10.4. Some properties of f_D

$$\int\cdots\int u_1^{b_1-1}\cdots u_n^{b_n-1}(1-u_1\cdots u_n)^{c-b_1-\cdots-b_n-1} \tag{1.10.13}$$
$$\times(1-u_1x_1-\cdots-u_nx_n)^{-a}\mathrm{d}u_1\wedge\cdots\wedge\mathrm{d}u_n$$
$$=\frac{\Gamma(b_1)\cdots\Gamma(b_n)\Gamma(c-b_1-\cdots-b_n)}{\Gamma(c)}f_D(a,b_1,\cdots,b_n;c;x_1,\cdots,x_n),\ \text{for}$$
$$0<u_j<1,\ j=1,\cdots,n,\ 0<u_1+\cdots+u_n<1,\ 0<x_1u_1+\cdots+x_nu_n<1,$$
$$\Re(b_j)>0,\ j=1,\cdots,n,\ \Re(c-b_1-\cdots-b_n)>0.$$

Open up $(1-u_1x_1-\cdots-u_nx_n)^{-a}$ by using a multinomial expansion and then integrate out $u_1,\cdots,u_n$ by using a type-1 Dirichlet integral of Subsection 1.6.1

$$\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux_1)^{-b_1}\cdots(1-ux_n)^{-b_n}\mathrm{d}u \tag{1.10.14}$$
$$=\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\ f_D(a,b_1,\cdots,b_n;c;x_1,\cdots,x_n)\ \text{ for }\Re(a)>0,\ \Re(c-a)>0.$$

Expand $(1-ux_j)^{-b_j}$, $j=1,\cdots,n$ by using binomial expansions and then integrate out u by using a type-1 beta integral to see the result.

$$\int_0^\infty\cdots\int_0^\infty t_1^{b_1-1}\cdots t_n^{b_n-1}\mathrm{e}^{-t_1-\cdots-t_n}{}_1F_1(a;c;x_1t_1+\cdots+x_nt_n)\mathrm{d}t_1\wedge\cdots\wedge\mathrm{d}t_n \tag{1.10.15}$$
$$=\Gamma(b_1)\cdots\Gamma(b_n)f_D(a,b_1,\cdots,b_n;c;x_1,\cdots,x_n),\ \text{for }\Re(b_j)>0,\ j=1,\cdots,n.$$

Expand ${}_1F_1$ as a series, then open up the general term with the help of a multinomial expansion for positive integral exponent, then integrate out $t_1,\cdots,t_n$ to see the result.

$$\frac{1}{(2\pi i)^n}\int\cdots\int\frac{\Gamma(a+t_1+\cdots+t_n)\Gamma(b_1+t_1)\cdots\Gamma(b_n+t_n)}{\Gamma(c+t_1+\cdots+t_n)} \tag{1.10.16}$$
$$\times\Gamma(-t_1)\cdots\Gamma(-t_n)(-x_1)^{t_1}\cdots(-x_n)^{t_n}\mathrm{d}t_1,\wedge\cdots\wedge\mathrm{d}t_n$$
$$=\frac{\Gamma(a)\Gamma(b_1)\cdots\Gamma(b_n)}{\Gamma(c)}f_D(a,b_1,\cdots,b_n;c;x_1,\cdots,x_n),i=\sqrt{-1}.$$

Follow through the same method of evaluation of the contour integrals as in f_A, f_B and f_C to see the result.

$$f_D(a,b_1,\cdots,b_n;c;x,\cdots,x)={}_2F_1(a,b_1+\cdots+b_n;c;x). \tag{1.10.17}$$

Use the integral representation in (1.10.14) and put $x_1=\cdots=x_n=x$ to see the result.

$$f_D(a, b_1, \cdots, b_n; c; 1, 1 \cdots, 1) = \frac{\Gamma(c)\Gamma(c - a - b_1 - \cdots - b_n)}{\Gamma(c - a)\Gamma(c - b_1 - \cdots - b_n)}. \tag{1.10.18}$$

Evaluate (1.10.17) at $x = 1$ to see the result.

There are other functions in the category of multivariable hypergeometric functions known as Humbert's functions, Kampé de Fériet functions and so on. These will not be discussed here. For a brief description of these, along with some of their properties, see for example Mathai (1993, 1997) and Srivasthava and Karlsson (1985).

Example 1.10.1. Show that

$$\begin{aligned} &f_A(a, b_1, \cdots, b_n; c_1; \cdots, c_n; x_1, \cdots, x_n) \qquad (1.10.19) \\ &= \sum_{m_1=0}^{\infty} \cdots, \sum_{m_{n-1}=0}^{\infty} \frac{(a)_{m_1+\cdots+m_{n-1}}(b)_{m_1} \cdots (b_n)_{m_n}}{(a)_{m_1} \cdots (c_n)_{m_n}} \\ &\times \frac{x_1^{m_1} \cdots x_{n-1}^{m_{n-1}}}{m_1! \cdots m_{n-1}!} {}_2F_1(a + m_1 + \cdots + m_{n-1}, b_n; c_n; x_n), |x_1| + \cdots + |x_n| < 1. \end{aligned}$$

Solution 1.10.1. This can be seen by summing up with respect to m_n by observing that $(a)_{m_1+\cdots+m_n} = (a + m_1 + \cdots + m_{n-1})_{m_n}$. Then the sum is the following:

$$\sum_{m_n=0}^{\infty} \frac{(a + m_1 + \cdots + m_{n-1})_{m_n}(b_n)_{m_n}}{(c_n)_{m_n}} \frac{x^{m_n}}{m_n!} = {}_2F_1(a + m_1 + \cdots + m_{n-1}, b_n; c_n; x_n).$$

Example 1.10.2. Show that

$$\begin{aligned} &\Gamma(a) f_C\Big(\frac{a}{2}, \frac{a+1}{2}; c_1, \cdots, c_n; x_1, \cdots, x_n\Big) \qquad (1.10.20) \\ &= \int_0^{\infty} t^{a-1} e^{-t} {}_0F_1\Big(; c_1; \frac{t^2 x_1}{4}\Big) \cdots {}_0F_1\Big(; c_n; \frac{t^2 x_n}{4}\Big)\, dt. \end{aligned}$$

Solution 1.10.2. Expand the ${}_0F_1$'s. Then the right side becomes,

$$\sum_{m_1=0}^{\infty} \cdots, \sum_{m_n=0}^{\infty} \frac{\int_0^{\infty} t^{a+2m_1+\cdots+2m_n-1}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} e^{-t} \frac{x_1^{m_1}}{4^{m_1}} \cdots \frac{x_n^{m_n}}{4^{m_n}} \frac{1}{m_1! \cdots m_n!} \, dt.$$

Integral over t yields

$$\int_0^\infty t^{a+2m_1+\cdots+2m_n-1}e^{-t}dt = \Gamma(a+2m_1+\cdots+2m_n).$$

Expanding $\Gamma(a+2m_1+\cdots+2m_n) = \Gamma[2(\frac{a}{2}+m_1+\cdots+m_n)]$ by using the duplication formula, we have,

$$\begin{aligned}\Gamma\Big[2\Big(\frac{a}{2}+m_1+\cdots+m_n\Big)\Big] &= \pi^{-\frac{1}{2}}2^{a+2m_1+\cdots+2m_n-1}\Gamma\Big(\frac{a}{2}+m_1+\cdots+m_n\Big)\\ &\quad\times\Gamma\Big(\frac{a+1}{2},m_1+\cdots+m_n\Big)\\ &= \pi^{-\frac{1}{2}}2^{a-1}\Gamma\Big(\frac{a}{2}\Big)\Gamma\Big(\frac{a+1}{2}\Big)\\ &\quad\times\Big(\frac{a}{2}\Big)_{m_1+\cdots+m_n}\Big(\frac{a+1}{2}\Big)_{m_1+\cdots+m_n}(4)^{m_1+\cdots+m_n}\\ &= \Gamma(a)\Big(\frac{a}{2}\Big)_{m_1+\cdots+m_n}\Big(\frac{a+1}{2}\Big)_{m_1+\cdots+m_n}(4)^{m_1+\cdots+m_n}\end{aligned}$$

(duplication formula is again applied on $\Gamma(a) = \Gamma[2(\frac{a}{2})]$). Now, substituting and interpreting as a f_C the result follows.

Example 1.10.3. Show that $f_B(a_1,\cdots,a_n,b_1,\cdots,b_n;c;x_1,\cdots,x_n)$

$$\begin{aligned}&= \frac{\Gamma(c)}{\Gamma(d_1)\cdots\Gamma(d_n)\Gamma(c-d_1-\cdots-d_n)}\int\cdots\int u_1^{d_1-1}\cdots u_n^{d_n-1} \qquad (1.10.21)\\ &\times(1-u_1-\cdots-u_n)^{c-d_1-\cdots-d_n-1}\\ &\times\ {}_2F_1(a_1,b_1;d_1;u_1x_1)\cdots{}_2F_1(a_n,b_n;d_n;u_nx_n)du_1\wedge\cdots\wedge du_n\end{aligned}$$

for $\Re(\mathrm{d}_j)>0,1,\cdots,n$, $\Re(c-d_1-\cdots-d_n)>0, |x_j|<1, j=1,\cdots,n.$

Solution 1.10.3. Expand the product of ${}_2F_1$'s first.

$$\begin{aligned}&{}_2F_1(a_1,b_1;d_1;u_1x_1)\cdots{}_2F_1(a_n,b_n;d_n;u_nx_n) \qquad (1.10.22)\\ &= \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(a_1)_{m_1}\cdots(a_n)_{m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(d_1)_{m_1}\cdots(d_n)_{m_n}}\frac{(u_1x_1)^{m_1}\cdots(u_nx_n)^{m_n}}{m_1!\cdots m_n!}.\end{aligned}$$

Now, evaluate the integral over $u_1,\cdots,u_n$ by using a type-1 Dirichlet integral.

$$\int \cdots \int u_1^{d_1+m_1-1} \cdots u_n^{d_n+m_n-1}(1-u_1-\cdots-u_n)^{c-d_1-\cdots-d_n-1} du_1 \wedge \cdots \wedge du_n$$
$$= \frac{\Gamma(d_1+m_1)\cdots\Gamma(d_n+m_n)\Gamma(c-d_1-\cdots-d_n)}{\Gamma(c+m_1+\cdots+m_n)}$$
$$= \frac{\Gamma(c-d_1-\cdots-d_n)\Gamma(d_1)\cdots\Gamma(d_n)}{\Gamma(c)} \frac{(d_1)_{m_1}\cdots(d_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \tag{1.10.23}$$

for $\Re(d_j) > 0$, $j = 1, \cdots, n$, $\Re(c - d_1 - \cdots - d_n) > 0$. Now, substituting (1.10.23) and (1.10.22) on the right side of (1.10.21) the result follows.

Exercises 1.10.

1.10.1. Establish the result in (1.10.5)

1.10.2. Establish the result in (1.10.6)

1.10.3. Establish the result in (1.10.7)

1.10.4. Establish the result in (1.10.8)

1.10.5. Establish the result in (1.10.9)

1.10.6. Establish the result in (1.10.10)

1.10.7. Establish the result in (1.10.11)

1.10.8. Establish the result in (1.10.12)

1.10.9. Establish the result in (1.10.13)

1.10.10. Establish the result in (1.10.14)

1.11. Special Functions as Solutions of Differential Equations and Applications

[*This section is based on the lectures of Professor P. N. Rathie of the Department of Statistics, University of Brasília, Brazil.*]

1.11.1. Introduction

Certain special functions occur often in fields like physics and engineering. We study these functions (Exponential to Mejer's G-functions) as solutions of differential equations because the behavior of a physical system is generally represented by a differential equation. A powerful method for solving differential equations is to assume a power series solution.

1.11.2. Lambert's W-function

Lambert's W-function (or Omega function), named after Johann Heinrich Lambert, is defined by

$$W(x)\mathrm{e}^{W(x)} = x \tag{1.11.1}$$

for $x \geq -\frac{1}{\mathrm{e}}$. Note that the logarithmic function is defined by $\mathrm{e}^{f(x)} = x$. For $-\frac{1}{\mathrm{e}} \leq x < 0$ there are two possible real values for $W(x)$ and for $x > 0$ there is only on possible real value. Generally, the branch satisfying $-1 \leq W(x)$ is denoted by $W_0(x)$ (the principal branch) and the branch satisfying $W(x) \leq -1$ by $W_{-1}(x)$.

One may note that

(a) $W_0(0) = 0$
(b) $W_0\left(-\frac{1}{\mathrm{e}}\right) = -1$
(c) $W(x)$ cannot be expressed in terms of elementary functions
(d) $W(x)$ satisfies the following differential equation

$$x(1 + W(x))\frac{\mathrm{d}W(x)}{\mathrm{d}x} = W(x), \text{ for } x \neq \frac{1}{\mathrm{e}} \tag{1.11.2}$$

(e) Using Lagrange inversion theorem, the Taylor series of $W_0(x)$ around zero is given by

$$W_0(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n, \tag{1.11.3}$$

with radius of convergence $\frac{1}{e}$ (Ratio test).

$W(x)$ has many applications in pure and applied mathematics, some of which we discuss here which are connected with differential equations.

Example 1.11.1. (Volterra equations for population growth)

$$\frac{dx}{dt} = ax(1-y), \quad \frac{dy}{dt} = -cy(1-x). \tag{1.11.4}$$

The analytic solution is a closed loop in the phase plane. If the upper branch is y^+ and the lower y^-, then

$$\begin{cases} y^+ = -W_{-1}\left(-\alpha x^{-c/a} e^{cx/a}\right) \\ \\ y^- = -W_0\left(-\alpha x^{-c/a} e^{cx/a}\right) \end{cases} \tag{1.11.5}$$

where α is an arbitrary constant.

Example 1.11.2. (Combustion problem) This problem involves solution of the differential equation

$$\frac{dy(t)}{dt} = y^2(t)(1-y(t)) \tag{1.11.6}$$

for $y(0) = \epsilon > 0, \ (0 < \epsilon \leq y < 1)$.

We have

$$\int \frac{dy(t)}{y^2(t)(1-y(t))} = \int dt + c$$

or

$$\int \left[\frac{1}{y^2(t)} + \frac{1}{y(t)(1-y(t))}\right] dy(t) = t + c$$

or

$$-\frac{1}{y(t)} + \int \left[\frac{1}{y(t)} + \frac{1}{1-y(t)}\right] dy(t) = t + c$$

or

$$-\frac{1}{y(t)} + \ln(y(t)) - \ln(1-y(t)) = t + c.$$

For $t = 0$, we have

$$-\frac{1}{\epsilon} + \ln(\epsilon) - \ln(1-\epsilon) = c.$$

Thus

$$-\frac{1}{y(t)} + \ln\frac{y(t)}{1-y(t)} = t - \frac{1}{\epsilon} + \ln\frac{\epsilon}{1-\epsilon} = f(t), \text{ say.}$$

This yields

$$\frac{1}{\frac{1}{y(t)} - 1} = \exp\left(\frac{1}{y(t)} + f(t) - 1 + 1\right)$$

That is,

$$e^{-1-f(t)} = \left(\frac{1}{y(t)} - 1\right)\exp\left(\frac{1}{y(t)} - 1\right).$$

Thus

$$\frac{1}{y(t)} - 1 = W\left(e^{-1-f(t)}\right)$$

giving

$$y(t) = \frac{1}{1 + W\left[\left(\frac{1}{\epsilon} - 1\right)\exp\left(\frac{1}{\epsilon} - 1 - t\right)\right]} \tag{1.11.7}$$

(Principal branch of W should be used).

Exercises 1.11.

1.11.1. (An enzyme kinetics problem) Leading order term in the solution of this problem satisfies the equation

$$\frac{ds_0(\tau)}{d\tau} = -\frac{(\sigma+1)s_0(\tau)}{\sigma s_0(\tau) + 1}.$$

Express $s_0(\tau)$ in terms of τ.

1.11.2. (Richard's equations for water movement in soil) Solve the following equation for $A(t)$,

$$\alpha A(t)\frac{\mathrm{d}A(t)}{\mathrm{d}t} = 1 - A(t).$$

1.11.3. (Navier-Stokes equations in parametric form)

For $x = pe^p$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = p$ with $y = \int W(x)\mathrm{d}x$ obtain y in terms of p.

1.11.4. Solve (1.11.4) to obtain (1.11.5).

1.11.3. Sine, cosine and exponential functions

The sine, cosine and the exponential functions are the most elementary special functions. The following example illustrates how sine and cosine functions are obtained as solutions of a differential equation in a mathematical physics problem.

Example 1.11.3. (Motion of an elastically bound particle in one dimension) The position x of a particle of mass m that moves under the influence of a force

$$F = -mw^2 x \tag{1.11.8}$$

is given at time t by

$$x = b\sin(wt + \phi) \tag{1.11.9}$$

where b and ϕ are constants. By Newton's second law of motion,

$$F = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}. \tag{1.11.10}$$

If the particle obeys *Hooke's law*, then (1.11.8) is valid with w as a constant. Thus (1.11.10) and (1.11.8) yield

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + w^2 x = 0. \tag{1.11.11}$$

Assuming the power series solution

$$x(t) = \sum_{n=0}^{\infty} a_n t^{s+n} \tag{1.11.12}$$

where the a_n are constant coefficients to be determined, we obtain

$$s(s-1)a_0 = 0 \tag{1.11.13}$$

$$(s+1)sa_1 = 0 \tag{1.11.14}$$

$$(s+n+2)(s+n+1)a_{n+2} + w^2 a_n = 0. \tag{1.11.15}$$

Solution 1.11.1. (1.11.13) and (1.11.14) are satisfied for $s = 0$. With this choice (1.11.14) gives

$$a_n = \frac{(-1)^{n/2} w^n}{n!} a_0, \text{ for } n \text{ even} \tag{1.11.16}$$

$$a_n = \frac{(-1)^{(n-1)/2} w^{n-1}}{n!} a_1 \text{ for } n \text{ odd.} \tag{1.11.17}$$

Thus, the general solution of (1.11.11) is given by

$$\begin{aligned} x(t) &= a_0 \sum_{\substack{n=0 \\ n=\text{even}}}^{\infty} \frac{(-1)^{n/2} w^n t^n}{n!} + a_1 \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{(-1)^{(n-1)/2} w^{n-1} t^n}{n!} \\ &= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k (wt)^{2k}}{(2k)!} + \frac{a_1}{w} \sum_{k=0}^{\infty} \frac{(-1)^k (wt)^{2k+1}}{(2k+1)!} \\ &= a_0 \cos(wt) + \frac{a_1}{w} \sin(wt). \end{aligned} \tag{1.11.18}$$

Thus the general solution of the second-order differential equation (1.11.11) is the sum of two linearly independent solutions. The constants a_0 and a_1 are determined from the initial conditions. For example, for $t = 0$, $x(0) = x_0$ and $v = v_0$ then $a_0 = x_0$ and $a_1 = v_0$. Thus

$$\begin{aligned} x(t) &= x_0 \cos(wt) + \frac{v_0}{w} \sin(wt) \\ &= b \sin(wt + \phi) \end{aligned} \tag{1.11.19}$$

with $b = \sqrt{x_0^2 + (v_0/w)^2}$ and $\phi = \tan^{-1}(x_0 w/v_0)$.

Exercises 1.11.

1.11.5. Find the solution of the differential equation

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} - y(x) = 0$$

1.11.6. Obtain the solution of Eq. (1.11.11) for the following cases

(1) $s = 1$,
(2) $s = -1$.

1.11.4. Linear second order differential equations

Any linear, second-order, homogeneous differential equation can be written in the form

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}u(z) + P(z)\frac{\mathrm{d}}{\mathrm{d}z}u(z) + Q(z)u(z) = 0. \tag{1.11.20}$$

Assuming $u(z)$ and $\frac{\mathrm{d}}{\mathrm{d}z}u(z)$ at $z = z_0$ and successive differentiations, we can get the Taylor series for $u(z)$ as

$$u(z) = \sum_{n=0}^{\infty} \frac{\frac{\mathrm{d}^n}{\mathrm{d}z^n}u(z_0)}{n!}(z - z_0)^n \tag{1.11.21}$$

If the series in (1.11.21) has a nonzero radius of convergence, then the solution exists. If $u(z)$ and $\frac{\mathrm{d}}{\mathrm{d}z}u(z)$ can be assigned arbitrary values at $z = z_0$, then we say that the point z_0 is an *ordinary point* of the differential equation (1.11.20), otherwise it is a *singular point*.

Consider the differential equation,

$$z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}u(z) + az\frac{\mathrm{d}}{\mathrm{d}z}u(z) + bu(z) = 0$$

where a and b are constants. For $z = 0$ we see that if $u(0)$ has any value other then zero, either $\frac{\mathrm{d}}{\mathrm{d}z}u(0)$ or $\frac{\mathrm{d}^2}{\mathrm{d}z^2}u(0)$ must be infinity and the Taylor series for $u(z)$ cannot be obtained around $z = 0$.

If $P(z)$ or $Q(z)$ has a singularity (not a branch point) at $z = z_0$ so that $\frac{d^2}{dz^2}u(z_0)$ cannot be obtained to construct the Taylor series of $u(z)$, then the differential equation has a *regular* singularity if and only if both $(z - z_0)P(z)$ and $(z - z_0)^2 Q(z)$ are analytic at z_0. Otherwise, the singularity is *irregular*.

1.11.5. Hypergeometric function

The Gauss hypergeometric differential equation is given by

$$z(1-z)\frac{d^2u}{dz^2} + [c - (a+b+1)z]\,\frac{du}{dz} - abu = 0 \tag{1.11.22}$$

or

$$[\delta(\delta + c - 1) - z(\delta + a)(\delta + b)]\,u = 0, \quad \text{where } \delta = z\frac{d}{dz}. \tag{1.11.23}$$

Assuming the solution

$$u(z) = \sum_{n=0}^{\infty} a_n z^{n+s} \tag{1.11.24}$$

we get from (1.11.22), the following relations,

$$s(s + c - 1)a_0 = 0 \tag{1.11.25}$$

and

$$a_{n+1} = \frac{(n+s)(n+s+a+b)+ab}{(n+s+1)(n+s+c)}a_n. \tag{1.11.26}$$

The trivial solution $u(z) = 0$ is obtained if we assume $a_0 = 0$. For the nontrivial solution we assume $a_0 \neq 0$. equation (1.11.25) yields $s = 0$ or $s = 1 - c$. For $s = 0$, the solution of (1.11.22) is given by

$$u(z) = a_0 \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n = a_0 \; {}_2F_1(a, b; c; z) \tag{1.11.27}$$

where $(a)_n$ stands for the Pochammer symbol.

Example 1.11.4. From (1.11.27), it is easy to see that

$${}_2F_1(1, b; b; z) = \sum_{n=0}^{\infty} z^n \quad \text{(geometric series)}$$

and

$${}_2F_1(-s, b; b; z) = (1 - z)^s \quad \text{(binomial expansion).}$$

The general solution (for $c \neq$ an integer) consists of two linearly independent solutions,

$$u(z) = c_1 u_1(z) + c_2 u_2(z) \tag{1.11.28}$$

where $u_1(z)$ is given by (1.11.27) and $u_2(z)$ corresponding to $s = -c + 1$ is given by

$$u_2(z) = z^{1-c} {}_2F_1(1 + a - c, 1 + b - c; 2 - c; z). \tag{1.11.29}$$

The arbitrary constants c_1 and c_2 are to be determined by the boundary conditions.

We may observe the following:

(1) The Gauss hypergeometric equation (1.11.22) has three singularities at 0, 1 and ∞.

(2) If c is and integer, $u_2(z)$ is not a new solution. For example, $c = 1$, $u_1(z) = u_2(z)$. If neither a nor b is zero or a negative integer, two linearly independent solutions are

$${}_2F_1(a, b; 1; z) \tag{1.11.30}$$

and (logarithmic solution)

$${}_2F_1(a, b; 1; z) = \ln(z) + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n z^n}{(n!)^2} \left[\sum_{i=1}^{n} \frac{1}{a + i - 1} + \sum_{i=1}^{n} \frac{1}{b + i - 1} - \sum_{i=1}^{n} \frac{2}{i} \right]. \tag{1.11.31}$$

Example 1.11.5. (Simple pendulum)

A simple pendulum consists of a point mass m attached to one end of a massless cord of length l, and the other end fixed at a point such that the system can swing freely under gravity. Let T_1 be the tension in the cord when it is inclined at an angle θ with the vertical. Then by Newton's second law, we have

$$T_1 - mg \cos\theta = \frac{mv^2}{l} \tag{1.11.32}$$

$$-mg \sin\theta = m \frac{\mathrm{d}^2}{\mathrm{d}t^2}(l\theta) \tag{1.11.33}$$

The equation (1.11.32), takes the following from

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l} \sin\theta = 0 \tag{1.11.34}$$

Multiplying (1.11.33) by $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - \frac{g}{l}\cos\theta\right] = 0.$$

which implies that

$$\frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - \frac{g}{l}\cos\theta,$$

is a constant. Let $\theta = \theta_0$ when $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$ (pendulum at rest). This gives the value of constant as $\frac{g}{l}\cos\theta_0$ so that

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}.$$

Integration gives the period of oscillation as

$$T = 2\pi\sqrt{\frac{l}{g}}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \sin^2\left(\frac{\theta_0}{2}\right)\right). \tag{1.11.35}$$

Thus, we see that the period of oscillation of a simple pendulum depends on the amplitude of the oscillation. Note that if the amplitude θ_0 is small, then the period is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Exercises 1.11.

1.11.7. Write down the general solution of the differential equation

$$z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}u(z) + \left(\frac{5}{4} - 2z\right)\frac{\mathrm{d}}{\mathrm{d}z}u(z) + \frac{3}{4}u(z) = 0.$$

1.11.8. Find the general solution of the differential equation

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x) + \mu\frac{\mathrm{d}}{\mathrm{d}x}y(x) + \lambda y(x) = 0$$

where λ and μ are constants. Discuss the conditions to be imposed on λ and μ.

1.11.9. Solve the differential equation

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x) + 2\frac{\mathrm{d}}{\mathrm{d}x}f(x) + xf(x) = 0$$

by series method. Is it possible to write the solution in terms of elementary functions?

1.11.10. Find the general solution of the differential equation

$$z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}u(z) + \mu(1-z)\frac{\mathrm{d}}{\mathrm{d}z}u(z) + \mu u(z) = 0,$$

where μ is not an integer. Is one of the solutions a polynomial?

1.11.11. Show that

$$y(z) = z^{-\alpha}\mathrm{e}^{-f(z)}{}_2F_1(a, b; c; h(z))$$

satisfies the second order differential equation:

$$\begin{aligned}
&\frac{h(h-1)}{(h')^2}y'' + \left\{\frac{h(h-1)}{(h')^3}\left(\frac{2\alpha h'}{z} + 2f'h' - h''\right) + \frac{(a+b+1)h-c}{h'}\right\}y' \\
&+\left\{\left(\frac{\alpha}{z} + f'\right)\left(\frac{(a+b+1)h-c}{h'}\right) + \frac{h(h-1)}{(h')^3}\left[\frac{\alpha(\alpha-1)h'}{z^2} + \frac{2\alpha f'h'}{z}\right.\right. \\
&\left.\left.+f''h' + (f')^2h' - \frac{\alpha h''}{z} - f'h''\right] + ab\right\}y = 0.
\end{aligned}$$

1.11.6. Confluent hypergeometric function

The confluent hypergeometric differential equation is

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) + (c-x)\frac{\mathrm{d}}{\mathrm{d}x}u(x) - au(x) = 0 \qquad (1.11.36)$$

or

$$[\delta(\delta + c - 1) - z(\delta + a)]\,u(x) = 0, \quad \delta = z\frac{\mathrm{d}}{\mathrm{d}z}. \qquad (1.11.37)$$

This can be obtained from Gauss hypergeometric equation (1.11.22) by taking $x = bz$ and then making $b \to \infty$. This equation has singularities at $x = 0$ and $x = \infty$. A merged (confluence) of the singularities of equation (1.11.22) at $z = 1$ and $z = \infty$ has occurred. The singularity at $x = 0$ is regular and at $x = \infty$, irregular.

To get the general solution of (1.11.36), substitute

$$u(x) = \sum_{k=0}^{\infty} a_k x^{k+s} \tag{1.11.38}$$

so that

$$s(s - 1 + c)a_0 = 0. \tag{1.11.39}$$

Thus the nontrivial general solution to (1.11.36) is given by

$$u(x) = A\,{}_1F_1(a; c; x) + Bx^{1-c}\,{}_1F_1(1 + a - c; 2 - c; x). \tag{1.11.40}$$

Exercises 1.11.

1.11.12. Show that

$$y = z^{-\alpha} e^{-f(z)}\,{}_1F_1(a; c; h(z))$$

satisfies the second order differential equation

$$hy'' + \left\{\frac{2\alpha h}{z} + 2f'h - \frac{hh''}{h'} - hh' + ch'\right\} y' + \left\{h'\left(\frac{\alpha}{z} + f'\right)(c - h) + h\left[\frac{\alpha(\alpha - 1)}{z^2} + \frac{2\alpha f'}{z} + f'' + (f')^2 - \frac{h''}{h'}\left(\frac{\alpha}{z} + f'\right)\right] - a(h')^2\right\} y = 0.$$

1.11.7. Hermite polynomials

The differential equation

$$\frac{d^2}{dy^2} f(y) - 2y\frac{d}{dy} f(y) + 2nf(y) = 0 \tag{1.11.41}$$

is known as Hermite's equation. For $z = y^2$, (1.11.41) takes the following from

$$z\frac{d^2}{dz^2} f(z) + \left(\frac{1}{2} - z\right)\frac{d}{dz} f(z) + \frac{n}{2} f(z) = 0. \tag{1.11.42}$$

This is of the form of confluent hypergeometric equation (1.11.36) yielding the two solutions for (1.11.41) as

$$\begin{aligned} f(y) &= \frac{(-1)^{-n/2} n!}{\left(\frac{n}{2}\right)!} {}_1F_1\left(-\frac{n}{2}; \frac{1}{2}; y^2\right) \text{ for } n \text{ even} \\ &\equiv H_n(y) \end{aligned} \tag{1.11.43}$$

and

$$\begin{aligned} f(y) &= \frac{(-1)^{(1-n)/2} 2n! y}{\left(\frac{n-1}{2}\right)!} {}_1F_1\left(-\frac{n-1}{2}; \frac{3}{2}; y^2\right) \text{ for } n \text{ odd} \\ &\equiv H_n(y) \end{aligned} \tag{1.11.44}$$

Exercises 1.11.

1.11.13. For what values of s the substitution $u(z) = z^s f(z)$ in

$$z^2\frac{d^2}{dz^2}u(z) - z^2\frac{d}{dz}u(z) + \frac{3}{16}u(z) = 0$$

result in confluent hypergeometric equation. Write the general solution for $u(z)$.

1.11.14. For what value of a the substitution $u(z) = e^{az} f(z)$ in

$$z\frac{d^2}{dz^2}u(z) + (1 - 5z)\frac{d}{dz}u(z) + 6zu(z) = 0$$

result in confluent hypergeometric equation. Find the solution.

1.11.15. Find the solution of the type $u(x) = v(x)w(x)$ for the differential equation

$$\frac{d^2}{dx^2}u(x) - \frac{d}{dx}u(x) - \frac{1}{4x^2}\left[a(a-2) + 2ax\right] u(x) = 0.$$

Choose $v(x)$ so that $w(x)$ satisfies the confluent hypergeometric equation. Find the general solution $u(x)$. Show that a particular solution reduces to $x^{a/2}e^x$.

1.11.8. Bessel functions

The differential equation

$$x^2 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + x\frac{\mathrm{d}u}{\mathrm{d}x} + (x^2 - \nu^2)u = 0 \tag{1.11.45}$$

(ν need not to be an integer) is called *Bessel equation*. There are two singularities at $x = 0$ (regular) and $x = \infty$ (irregular).

The general solution of (1.11.45), when ν is not an integer, is given by

$$u(x) = \mathrm{e}^{-ix}\left[A_\nu x^\nu {}_1F_1\left(\nu + \frac{1}{2}; 2\nu + 1; 2ix\right) + B_\nu x^{-\nu} {}_1F_1\left(-\nu + \frac{1}{2}; -2\nu + 1; 2ix\right)\right] \tag{1.11.46}$$

where A_ν and B_ν are arbitrary constants. Using the relation

$$J_\nu(x) = \frac{\mathrm{e}^{-ix}(x/2)^\nu}{\Gamma(\nu + 1)} {}_1F_1\left(\nu + \frac{1}{2}; 2\nu + 1; 2ix\right), \tag{1.11.47}$$

where $J_\nu(x)$ is Bessel function of the first kind, the solution (1.11.42) may be written as

$$u(x) = a_\nu J_\nu(x) + b_\nu J_{-\nu}(x) \tag{1.11.48}$$

where $a_\nu = 2^\nu \Gamma(\nu + 1)A_\nu$, $b_\nu = 2^{-\nu}\Gamma(-\nu + 1)B_\nu$. It may be noted that $J_\nu(x)$ is regular at $x = 0$, whereas $J_{-\nu}(x)$ is irregular at $x = 0$. The function

$$N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \tag{1.11.49}$$

which is known as Bessel function of the second kind (or Newman's function) is also a solution of (1.11.45) and is linearly independent of $J_\nu(x)$. The function $N_\nu(x)$ is irregular at $x = 0$. Thus the general solution of (1.11.45) may also be written as

$$u(x) = aJ_\nu(x) + bN_\nu(x) \tag{1.11.50}$$

where a and b are arbitrary constants. Two functions frequently encountered in physical applications are

$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x) \tag{1.11.51}$$

and

$$H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x) \tag{1.11.52}$$

which are known as Bessel functions of the third kind (or Hankel functions).

If ν is zero or a positive integer, then the two independent solutions of (1.11.45) are $J_\nu(x)$ and a logarithmic solution which has a complicated expression.

Exercises 1.11.

1.11.16. Let $u(x) = \sum_{k=0}^{\infty} c_k x^{k+s}$ be the solution of the Bessel equation

$$x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + (x^2 - \nu^2)u = 0.$$

Show that for $k > 1$ the coefficient c_k satisfies

$$\left\{(k+s)^2 - \nu^2\right\} c_k + c_{k-2} = 0.$$

For $\nu^2 \neq \frac{1}{4}$, verify that, if $c_0 \neq 0$, then $c_1 = 0$ and vice versa. Obtain the solutions of the Bessel equation.

1.11.17. With the change of dependent variable $w(x) = e^{bx} f(x)$ show that the differential equation

$$x\frac{d^2}{dx^2}w(x) + 2\lambda\frac{d}{dx}w(x) + xw(x) = 0$$

can be transformed into the confluent hypergeometric equation. Write down the general solution expressed in terms of Bessel functions.

1.11.18. Transform the differential equation

$$\frac{d^2}{dx^2}u(x) + \lambda x^s u(x) = 0$$

(λ = constant, s = a real positive number) into Bessel equation using $u(x) = x^p f(x)$ and $z = bx^q$. Find the values of b, p and q. Write down the differential equation for f as a function of z. Find the general solution to the original equation.

1.11.9. Laguerre polynomial

The Laguerre polynomial

$$L_n^{(\alpha)}(x) = \sum_{i=0}^{n} \frac{(1+\alpha)_n(-x)^i}{i!(n-i)!(1+\alpha)_i} \tag{1.11.53}$$

is a solution of the differential equation

$$x\frac{d^2y}{dx^2} + (1+\alpha-x)\frac{d}{dx}y + ny = 0 \tag{1.11.54}$$

Laguerre polynomial (1.11.53) is connected to confluent hypergeometric function by the relation

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_1F_1(-n; 1+\alpha; x). \tag{1.11.55}$$

1.11.10. Legendre polynomial

The Legendre's equation is

$$(1-x^2)\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}f}{\mathrm{d}x} + l(l+1)f = 0, \quad l = 0, 1, \dots \tag{1.11.56}$$

which has its solution as

$$P_l(x) = {}_2F_1\left(-l, l+1; 1; \frac{1}{2}(1-x)\right) \tag{1.11.57}$$

which is a polynomial of order l.

Exercises 1.11.

1.11.19. Show that the Legendre's equation

$$(1-x^2)\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}f}{\mathrm{d}x} + n(n+1)f = 0$$

has a solution

$$f(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!\, x^{n-2k}}{2^n k!(n-2k)!(n-k)!}$$

where n is either even or odd.

1.11.20. Show that the differential equation

$$(1-x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + n^2 y = 0$$

has three regular singularities.

1.11.21. Obtain the general solution to the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + n(n+2)y = 0$$

in terms of hypergeometric functions. Show that one of the solutions is a polynomial if n is an integer ($n \neq -1$).

1.11.11. Generalized hypergeometric function

The homogeneous linear differential equation

$$\left[\delta \prod_{j=1}^{q}(\delta + b_j - 1) - z\prod_{i=1}^{p}(\delta + a_i)\right] u(z) = 0 \tag{1.11.58}$$

where $\delta = z\frac{d}{dz}$,

(a) is of order $\max(p, q+1)$,

(b) has singularities at $z = 0$ (regular) and $z = \infty$ (irregular) for $p < q + 1$,

(c) has regular singularities at $z = 0$, 1 and ∞ when $p = q + 1$.

The $q + 1$ linearly independent solutions of (1.11.58) for $p \le q + 1$ near $z = 0$ when no two b_j's differ by an integer or zero, and no b_j is a negative integer or zero, are given by

$$u_h(z) = A_h z^{1-b_h}\, {}_pF_q(1 + a_1 - b_h, \ldots, 1 + a_p - b_h; 1 + b_0 - b_h, \ldots, 1 + b_q - b_h; z) \tag{1.11.59}$$

for $h = 0, 1, \ldots, q$, $b_0 = 1$, with the term $1 + b_j - b_j$, $j = 0, 1, \ldots, q$ omitted where A_h, $h = 0, 1, \ldots, q$ are arbitrary constants. Thus

$$\begin{cases} u_0 = A_0 \;\; {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \\ u_1 = A_1 \;\; z^{1-b_1}\, {}_pF_q(1 + a_1 - b_1, \ldots, 1 + a_p - b_1; 2 - b_1, 1 + b_2 - b_1, \ldots, 1 + b_q - b_1; z) \\ u_2 = A_2 \;\; z^{1-b_2}\, {}_pF_q(1 + a_1 - b_2, \ldots, 1 + a_p - b_2; 1 + b_1 - b_2, 2 - b_2, 1 + b_3 - b_2, \ldots, 1 + b_q - b_2; z) \\ \text{etc} \end{cases} \tag{1.11.60}$$

Solution 1.11.2. Substitution of

$$u(z) = \sum_{n=0}^{\infty} c_n z^{n+s} \tag{1.11.61}$$

in (1.11.58) yields

$$\sum_{n=0}^{\infty} c_n \left\{ (s+n) \prod_{j=1}^{q} (s+n+b_j-1) z^n - \prod_{i=1}^{p} (s+n+a_i) z^{n+1} \right\} = 0. \tag{1.11.62}$$

Equating the coefficient of c_0 to zero gives the indicial equation roots as $s_h = 1 - b_h$, $h = 0, 1, \ldots, q$. For the root s_h, we find from (1.11.62) that

$$c_n = \frac{\prod_{i=1}^{p} (s_h + \alpha_i) c_0}{(s_h+1)_n \prod_{j=1}^{n} (s_h + b_j)_n} \tag{1.11.63}$$

Thus (1.11.63) and (1.11.61) yields (1.11.60). If $p \geq q+1$ and no two of the a_i's differ by an integer or zero, there are p linearly independent solutions of the equation (1.11.58) near $z = \infty$:

$$v_h(t) = B_h z^{-a_h} {}_{q+1}F_{p-1}(1+a_h-b_1, \ldots, 1+a_h-b_q; 1+a_h-a_1, \ldots, 1+a_h-a_p; (-1)^{q+1-p}/z),$$

$h = 1, 2, \ldots, p$, where B_h, $h = 1, \ldots, p$, are arbitrary constants.

1.11.12. G-function

The G-function

$$y(z) = G_{p,q}^{m,n} \left[z \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \right] \tag{1.11.64}$$

satisfies the homogeneous linear differential equation

$$\left[(-1)^{m+n-p} z \prod_{i=1}^{p} (\delta - a_i + 1) - \prod_{j=1}^{q} (\delta - b_j) \right] y(z) = 0 \tag{1.11.65}$$

where $\delta = z\frac{d}{dz}$. This equation

(a) is of order $\max(p, q)$,

(b) has singularities at $z = 0$ (regular), $z = \infty$ (irregular) when $p < q$,

(c) has regular singularities at $z = 0$, ∞ and $(-1)^{m+n-p}$ if $p = q$.

In view of

$$G_{p,q}^{m,n}\left[z\left|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right.\right]=G_{q,p}^{n,m}\left[\frac{1}{z}\left|\begin{matrix}1-b_1,\dots,1-b_q\\ 1-a_1,\dots,1-a_p\end{matrix}\right.\right],\quad -\arg z=\arg\left(\frac{1}{z}\right) \tag{1.11.66}$$

it is enough to consider $p < q$. The q functions

$$y_h(z)=A_h \mathrm{e}^{i\pi(m+n-p+1)b_h} G_{p,q}^{1,p}\left[z\mathrm{e}^{-i\pi(m+n-p+1)}\left|\begin{matrix}a_1,\dots,a_p\\ b_h,b_1,\dots,b_{h-1},b_{h+1},\dots,b_q\end{matrix}\right.\right], \tag{1.11.67}$$

$h = 1, 2, \dots, q$ for A_h arbitrary constant, form linearly independent solutions for (1.11.65) around $z = 0$ provided that no two of $b_j,\ j = 1, \dots, m$, differ by an integer or zero. Equation (1.11.67) may be written as

$$y_h(z)=\frac{\prod_{i=1}^{p}\Gamma(1+b_h-a_i)}{\prod_{j=1}^{q}\Gamma(1+b_h-b_j)}A_h z^{b_h}\times \tag{1.11.68}$$
$$\times\, {}_pF_{q-1}(1+b_h-a_1,\dots,1+b_h-a_p;1+b_h-b_1,\dots,1+b_h-b_q;(-1)^{m+n-p}z),$$

where $p \le q-1$ or $p = q$ and $|z| < 1$ and $1 + b_h - b_h$ term is omitted. When two or more b_j's differ by an integer or zero, the corresponding independent solution may involve log, psi and/or zeta functions. The solution of (1.11.65) in the neighborhood of $z = \infty$ (irregular singularity) are rather lengthy to obtain.

Exercises 1.11.

1.11.22. Show that $u(z) = {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z)$ satisfies eq. (1.11.58).

1.11.23. Show that (1.11.64) satisfies (1.11.65).

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