CHAPTER 3

FRACTIONAL CALCULUS AND FRACTIONAL DIFFERENTIAL EQUATIONS

[*This chapter is based on the lectures of Professor R.K. Saxena of Jai Narain Vyas University, Jodhpur, Rajasthan, India.*]

3.0. Introduction

This section deals with certain properties of fractional calculus associated with Laplace and Mellin transforms. Composition relations between Riemann-Liouville fractional calculus and generalized Mittag-Leffler functions are presented. Applications of fractional calculus in the solution of differential and integral equations of fractional order are demonstrated. This study will bring the reader to the research level.

3.1. Laplace Transform of the Fractional Integral

3.1.1. Laplace transform

Notation 3.1.1. $F(s) = L\{f(t); s\} = (Lf)(s)$: Laplace transform of $f(t)$ with parameter s.

Notation 3.1.2. $L^{-1}\{f(s); t\}$: Inverse Laplace transform

Definition 3.1.1. The Laplace transform of a function $f(t)$, denoted by $F(s)$, is defined by the equation

$$F(s) = (Lf)(s) = L\{f(t); s\} = \int_0^\infty e^{-st} f(t)dt, \qquad (3.1.1)$$

where $\Re(s) > 0$, which may be symbolically written as

$$F(s) = L\{f(t); s\} \text{ or } f(t) = L^{-1}\{F(s); t\},$$

provided that the function $f(t)$ is continuous for $t \geq 0$, it being tacitly assumed that the integral in (3.1.1) exists.

Example 3.1.1. Prove that

$$L^{-1}\{s^{-\rho}\} = \frac{t^{\rho-1}}{\Gamma(\rho)}, \quad \Re(s) > 0, \quad \Re(\rho) > 0. \tag{3.1.2}$$

It follows from the Laplace integral

$$\int_0^{\infty} e^{-st} t^{\rho-1} dt = \frac{\Gamma(s)}{s^{\rho}}, \quad \Re(s) > 0, \quad \Re(\rho) > 0. \tag{3.1.3}$$

Example 3.1.2. Find the inverse Laplace transform of $\frac{F(s)}{a+s^{\alpha}}$; $a, \alpha > 0$; where $\Re(s) > 0, F(s) = L\{f(t); s\}$.

Solution 3.1.1. Let

$$G(s) = \frac{1}{a + s^{\alpha}} = \sum_{r=0}^{\infty} (-a)^r s^{-\alpha-\alpha r}, \quad |\frac{a}{s^{\alpha}}| < 1.$$

Therefore,

$$\begin{aligned} L^{-1}\{G(s)\} = g(t) &= L^{-1}\left\{\sum_{r=0}^{\infty} (-a)^r s^{-\alpha-\alpha r}\right\} \\ &= t^{\alpha-1} E_{\alpha,\alpha}(-at^{\alpha}). \end{aligned} \tag{3.1.4}$$

Application of convolution theorem of Laplace transform yields the result

$$L^{-1}\left\{\frac{F(s)}{a + s^{\alpha}}; t\right\} = \int_0^x (x - t)^{\alpha-1} E_{\alpha,\alpha}(-a(x - t)^{\alpha}) \, f(t) dt \tag{3.1.5}$$

where $\Re(\alpha) > 0$.

3.1.2. Laplace transform of the fractional integral

We have

$$ {}_0I_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt, \tag{3.1.6}$$

where $\Re(\nu) > 0$.
Application of convolution theorem of the Laplace transform gives

$$\begin{aligned} L\{{}_0I_x^{-\nu} f(x); s\} &= L\left\{\frac{t^{\nu-1}}{\Gamma(\nu)}\right\} L\{f(t); s\} \\ &= s^{-\nu} F(s), \end{aligned} \tag{3.1.7}$$

where $\Re(s) > 0$, $\Re(\nu) > 0$.

3.1.3. Laplace transform of the fractional derivative

If $n \in \mathbb{N}$, then by the theory of the Laplace transform, we know that

$$L\left\{\frac{\mathrm{d}^n}{\mathrm{d}x^n} f; s\right\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0+) \tag{3.1.8}$$

$$= s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0+), \ (n-1 \le \alpha < n) \tag{3.1.9}$$

where $\Re(s) > 0$ and $F(s)$ is the Laplace transform of $f(t)$.
By virtue of the definition of the derivative, we find that

$$L\{{}_0D_x^{\alpha} f; s\} = L\left\{\frac{\mathrm{d}^n}{\mathrm{d}x^n} {}_0I_x^{n-\alpha} f; s\right\}$$

$$= s^n L\left\{{}_0I_x^{n-\alpha} f; s\right\} - \sum_{k=0}^{n-1} s^k \frac{\mathrm{d}^{n-k-1}}{\mathrm{d}x^{n-k-1}}\, {}_0I_x^{n-\alpha} f(0+)$$

$$= s^\alpha F(s) - \sum_{k=0}^{n-1} s^k D^{\alpha-k-1} f(0+), \left(D = \frac{\mathrm{d}}{\mathrm{d}x}\right) \tag{3.1.10}$$

$$= s^\alpha F(s) - \sum_{k=1}^{n} s^{k-1} D^{\alpha-k} f(0+) \tag{3.1.11}$$

where $\Re(s) > 0$.

3.1.4. Laplace transform of Caputo derivative

Notation 3.1.3. $\underset{0}{C}\, {}_aD_x^\alpha$

Definition 3.1.2. The Caputo derivative of a casual function $f(t)$ (that is $f(t) = 0$ for $t < 0$) with $\alpha > 0$ was defined by Caputo (1969) in the form

$$\underset{0}{C}\, {}_aD_x^\alpha f(x) = {}_aI_x^{n-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) = {}_aD_t^{-(n-\alpha)} f^n(t) \tag{3.1.12}$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^n(t)\mathrm{d}t, (n-1 < \alpha < n) \tag{3.1.13}$$

where $n \in \mathbb{N}$.

From (3.1.7) and (3.1.13), it follows that

$$L\{\underset{0}{C}\, {}_0D_t^\alpha f(t); s\} = s^{-(n-\alpha)} L\{f^{(n)}(t)\}. \tag{3.1.14}$$

On using (3.1.8), we see that

$$L\{\underset{0}{C}\, {}_0D_t^\alpha f(t); s\} = s^{-(n-\alpha)} \left[s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0+) \right]$$

$$= s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0+), \quad (n-1 < \alpha \le n), \tag{3.1.15}$$

where $\Re(s) > 0$ and $\Re(\alpha) > 0$.

Note 3.1.1. From (3.1.12), it can be seen that

$$ {}_0^C\,{}_0D_t^{\alpha} A = 0, \text{ where } A \text{ is a constant,} $$

whereas the Riemann-Liouville derivative

$$ {}_0D_t^{\alpha} A = \frac{At^{-\alpha}}{\Gamma(1-\alpha)}, \quad (\alpha \neq 1, 2, \cdots), \tag{3.1.16} $$

which is a remarkable result.

Exercises 3.1.

3.1.1. Prove that

$$ ({}_0I_x^{-\nu} f)(x) = L^{-1} x^{-\nu} L\{f(x)\}, \tag{3.1.17} $$

where $\Re(\nu) > 0$.

3.1.2. Prove that

$$ ({}_xW_{\infty}^{\nu} L f)(x) = L x^{-\nu} L^{-1} f(x), \tag{3.1.18} $$

where $\Re(\nu) > 0$.

3.1.3. Prove that the solution of Abel integral equation of the second kind

$$ \phi(x) - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{\phi(t)\mathrm{d}t}{(x-t)^{1-\alpha}} = f(x), \; 0 < x < 1 $$

$\alpha > 0$, is given by

$$ \phi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x E_{\alpha}[\lambda(x-t)^{\alpha}] \; f(t) \; \mathrm{d}t, \tag{3.1.19} $$

where $E_{\alpha}(x)$ is the Mittag-Leffler function defined by equation (3.5.1).

3.1.4. Show that

$$ \frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{E_{\alpha}(\lambda t^{\alpha})}{(x-t)^{1-\alpha}} \mathrm{d}t = E_{\alpha}(\lambda x^{\alpha}) - 1, \; \alpha > 0. \tag{3.1.20} $$

3.2. Mellin Transform of the Fractional Integrals and the Fractional Derivatives

3.2.1. Mellin transform

Notation 3.2.1. $m\{f(x);\ s\}$, $f^*(s)$, the Mellin transform

Notation 3.2.2. $m^{-1}\{f^*(s);\ x\}$, Inverse Mellin transform

Definition 3.2.1. The Mellin transform of a function $f(x)$, denoted by $f^*(s)$, is defined by

$$f^*(s) = m\{f(x);\ s\} = \int_0^\infty x^{s-1}\ f(x)\mathrm{d}x,\ \ x > 0. \tag{3.2.1}$$

The inverse Mellin transform is given by the contour integral

$$f(x) = m^{-1}\{f^*(s);\ x\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)x^{-s}\mathrm{d}s,\ \ i = \sqrt{-1} \tag{3.2.2}$$

where γ is real.

3.2.2. Mellin transform of the fractional integral

Theorem 3.2.1. *The following result holds true.*

$$m({}_0I_x^\alpha\ f)(s) = \frac{\Gamma(1-\alpha-s)}{\Gamma(1-\alpha)}f^*(s+\alpha), \tag{3.2.3}$$

where $\Re(\alpha) > 0$ *and* $\Re(\alpha + s) < 1$.

Proof 3.2.1. We have

$$\begin{aligned} m({}_0I_x^\alpha\ f)(s) &= \int_0^\infty z^{s-1}\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}f(t)\ \mathrm{d}t\mathrm{d}z \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty f(t)\ \mathrm{d}t\int_t^\infty z^{s-1}(z-t)^{\alpha-1}\ \mathrm{d}z \end{aligned} \tag{3.2.4}$$

on setting $z = \frac{t}{u}$, the z-integral becomes

$$t^{\alpha+s-1}\int_0^1 u^{-\alpha-s}(1-u)^{\alpha-1}\mathrm{d}u = t^{\alpha+s-1}B(\alpha, 1-\alpha-s), \tag{3.2.5}$$

where $\Re(\alpha) > 0,\ \ \Re(\alpha+s) < 1$. Putting the above value of z-integral, the result follows.

Similarly we can establish

Theorem 3.2.2. *The following result holds true.*

$$\begin{aligned} m({}_xI_\infty^\alpha\ f)(s) &= \frac{\Gamma(s)}{\Gamma(s+\alpha)}\ m\,\{t^\alpha f(t);\ s\} \\ &= \frac{\Gamma(s)}{\Gamma(s+\alpha)} f^*(s+\alpha), \end{aligned} \tag{3.2.6}$$

where $\Re(\alpha) > 0,\ \ \Re(s) > 0$.

Note 3.2.1. If we set $f(x) = x^{-\alpha}\phi(x)$, then using the property of the Mellin transform

$$x^\alpha\phi(x) \leftrightarrow \phi^*(s+\alpha), \tag{3.2.7}$$

the results (3.2.3) and (3.2.6) become

$$({}_0I_x^\alpha\ x^{-\alpha} f(x))(s) = \frac{\Gamma(1-\alpha-s)}{\Gamma(1-s)}\ f^*(s), \tag{3.2.8}$$

where $\Re(\alpha) > 0,\ \ \Re(\alpha+s) < 1$ and

$$({}_xI_\infty^\alpha\ x^{-\alpha} f(x))(s) = \frac{\Gamma(s)}{\Gamma(s+\alpha)}\ f^*(s), \tag{3.2.9}$$

where $\Re(\alpha) > 0$, and $\Re(s) > 0$, respectively.

3.2.3. Mellin transform of the fractional derivative

Theorem 3.2.3. *If* $n \in \mathbb{N}$, *then*

$$m\{f^{(n)}(t);\ (s)\} = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} m\{f(t);\ s-n\}, \tag{3.2.10}$$

where $\Re(s) > 0, \Re(s-n) > 0.$

Proof 3.2.2. Integrate by parts and using the definition of the Mellin transform, the result follows.

Example 3.2.1. Find the Mellin transform of the fractional derivative.

Solution 3.2.1. We have

$$ {}_0D_x^{\alpha}\ f = {}_0D_x^{n}\ {}_0D_x^{\alpha-n}\ f = {}_0D_x^{n}\ {}_0I_x^{n-\alpha}\ f. \tag{3.2.11}$$

Therefore,

$$m({}_0D_x^{\alpha}\ f)(s) = \frac{(-1)^n\ \Gamma(s)}{\Gamma(s-n)}\ m\{{}_0I_x^{n-\alpha}\ f\}(s-n), (n-1 \le \Re(\alpha) < n) \tag{3.2.12}$$

$$= \frac{(-1)^n\ \Gamma(s)\Gamma(1-(s-\alpha))}{\Gamma(s-n)\Gamma(1-s+n)}\ m\{f(t); s-\alpha\}, \tag{3.2.13}$$

where $\Re(s) > 0, \Re(s) < 1 + \Re(\alpha).$

Remark 3.2.1. An alternative form of (3.2.13) is given in Exercise 3.2.2.

Exercises 3.2.

3.2.1 Prove Theorem 3.2.2.

3.2.2 Prove that the Mellin transform of fractional derivative is given by

$$m({}_0D_x^{\alpha}\ f)(s) = \frac{(-1)^n\ \Gamma(s)\sin[\pi(s-n)]}{\Gamma(s-\alpha)\sin[\pi(s-\alpha)]} m\{f(t); s-\alpha\}, \tag{3.2.14}$$

where $\Re(s) > 0, \Re(\alpha - s) > -1.$

3.2.3 Find the Mellin transform of $(1+x^a)^{-b}; a, b > 0.$

3.3. Kober Operators

Kober operators are the generalization of Riemann - Liouville and Weyl operators. These operators have been used by many authors in deriving the solution of single, dual and triple integral equations possessing special functions of mathematical physics, as their kernels.

Notation 3.3.1. Kober operator of the first kind

$$\mathbb{I}[f(x)],\ \mathbb{I}[\alpha,\eta : f(x)],\ \mathbb{I}(\alpha,\eta)f(x), E_{0,x}^{\alpha,\eta}\ f,\ \mathbb{I}_x^{n,\alpha}\ f.$$

Notation 3.3.2. Kober operator of the second kind

$$\mathbb{R}[f(x)],\ \mathbb{R}[\alpha,\zeta : f(x)],\ \mathbb{R}(\alpha,\zeta)f(x), K_{x,\infty}^{\alpha,\zeta}\ f, K_x^{\zeta,\alpha}\ f.$$

Definition 3.3.1.

$$\begin{aligned}\mathbb{I}[f(x)] = \mathbb{I}[\alpha,\eta : f(x)] = \mathbb{I}(\alpha,\eta)f(x) = E_{0,x}^{\alpha,\eta}f \\ = \mathbb{I}_x^{\eta,\alpha}\ f = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}t^{\eta}f(t)\mathrm{d}t,\end{aligned} \tag{3.3.1}$$

where $\Re(\alpha) > 0$.

Definition 3.3.2.

$$\begin{aligned}\mathbb{R}[f(x)] = \mathbb{R}[\alpha,\zeta : f(x)] = \mathbb{R}(\alpha,\zeta)f(x) = K_{x,\infty}^{\alpha,\zeta}f \\ = K_x^{\zeta,\alpha}\ f = \frac{x^{\zeta}}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1}t^{-\zeta-\alpha}f(t)\mathrm{d}t,\end{aligned} \tag{3.3.2}$$

where $\Re(\alpha) > 0$.

(3.3.1) and (3.3.2) hold true under the following conditions:

$$f \in L_p(0,\infty), \Re(\alpha) > 0, \Re(\eta) > -\frac{1}{q}, \Re(\zeta) > -\frac{1}{p}, \frac{1}{p} + \frac{1}{q} = 1, p \geq 1.$$

When $\eta = 0$, (3.3.1) reduces to Riemann - Liouville operator. That is,

$$I_x^{0,\alpha} f = x^{-\alpha}\ {}_0I_x^{\alpha} f. \tag{3.3.3}$$

For $\zeta = 0$, (3.3.2) yields the Weyl operator of $t^{-\alpha}f(t)$. That is,

$$K_x^{0,\alpha} f = {}_xW_\infty^\alpha \, t^{-\alpha} f(t). \tag{3.3.4}$$

Theorem 3.3.1. *[Kober (1940)].*

If $\Re(\alpha) > 0, \Re(\eta - s) > -1, f \in L_p(o,\infty), 1 \le p \le 2$ (or $f \in M_p(o,\infty)$, a subspace of $L_p(o,\infty)$ and $p > 2$), $\Re(\eta) > -\frac{1}{q}, \frac{1}{p} + \frac{1}{q} = 1$, then there holds the formula

$$m\{\mathbb{I}(\alpha,\eta) f\}(s) = \frac{\Gamma(1+\eta-s)}{\Gamma(\alpha+\eta+1-s)} m\{f(x);s\}. \tag{3.3.5}$$

Proof 3.3.1. It is similar to the proof of Theorem 3.2.1.

In a similar manner, we can establish

Theorem 3.3.2. *[Kober (1940)].*

If $\Re(\alpha) > 0, \Re(s+\zeta) > 0, f \in L_p(o,\infty), 1 \le p \le 2$ (or $f \in M_p(o,\infty)$, a subspace of $L_p(o,\infty)$ and $p > 2$)

$$\Re(\zeta) > -\frac{1}{p}, \frac{1}{p} + \frac{1}{q} = 1,$$

then,

$$m\{\Re(\alpha,\zeta) f\}(s) = \frac{\Gamma(\zeta+s)}{\Gamma(\alpha+\zeta+s)} m\{f(x);s\}. \tag{3.3.6}$$

Semigroup property of the Kober operators has been given in the form of

Theorem 3.3.3. *If $f \in L_p(o,\infty), g \in L_q(o,\infty), \frac{1}{p} + \frac{1}{q} = 1, \Re(\eta) > -\frac{1}{q}, \Re(\zeta) > -\frac{1}{p}, 1 \le p \le 2$, (or $f \in M_p(o,\infty)$, a subspace of $L_p(o,\infty)$ and $p > 2$), then*

$$\int_0^\infty g(x)(\mathbb{I}(\alpha,\eta : f))(x)\mathrm{d}x = \int_0^\infty f(x)(\mathbb{R}(\alpha,\zeta : g))(x)\mathrm{d}x. \tag{3.3.7}$$

Proof 3.3.2. Interchange the order of integration.

Remark 3.3.1. Operators defined by (3.3.1.) and (3.3.2) are also called Erdélyi-Kober operators.

Exercises 3.3.

3.3.1 Prove theorem 3.3.1.

3.3.2 For the modified Erdélyi-Kober operators, defined by the following equations for $m > 0$:

$$\begin{aligned}\mathbb{I}(\alpha,\eta : m) f(x) &= \mathbb{I}(f(x) : \alpha,\eta,m)\\ &= \frac{m}{\Gamma(\alpha)} x^{-\eta-m\alpha+m-1}\int_o^x t^{\eta}(x^m - t^m)^{\alpha-1} f(t)\mathrm{d}t, \qquad (3.3.8)\end{aligned}$$

and

$$\begin{aligned}\mathbb{R}(\alpha,\zeta : m) f(x) &= \mathbb{R}(f(x) : \alpha,\zeta,m)\\ &= \frac{mx^{\zeta}}{\Gamma(\alpha)}\int_x^{\infty} t^{-\zeta-m\alpha+m-1}(t^m - x^m)^{\alpha-1} f(t)\mathrm{d}t, \qquad (3.3.9)\end{aligned}$$

where $f \in L_p(0,\infty), \Re(\alpha) > 0, \Re(\eta) > -\frac{1}{q}, \Re(\zeta) > -\frac{1}{p}, \frac{1}{p} + \frac{1}{q} = 1$, find the Mellin transforms of (i) $\mathbb{I}(\alpha,\eta : m) f(x)$ and (ii) $\mathbb{R}(\alpha,\zeta : m) f(x)$, giving the conditions of validity.

3.3.3 For the operators defined by (3.3.8) and (3.3.9.), show that

$$\int_0^{\infty} \mathbb{R}(f(x) : \alpha,\eta,m) g(x)\mathrm{d}x = \int_0^{\infty} f(x)\mathbb{I}(g(x) : \alpha,\eta,m)\mathrm{d}x, \qquad (3.3.10)$$

where the parameters α,η,m are the same in both the operators $\mathbb{I}$ and $\mathbb{R}$. Give conditions of validity of (3.3.10).

3.3.4 For the Erdélyi-Kober operator, defined by

$$I_{\eta,\alpha} f(x) = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)}\int_0^x (x^2 - t^2)^{\alpha-1} t^{2\eta+1} f(t)\mathrm{d}t, \qquad (3.3.11)$$

where $\Re(\alpha) > 0$, establish the following results (Sneddon (1975)):

$$\text{(i)} \quad I_{\eta,\alpha} x^{2\beta} f(x) = x^{2\beta} I_{\eta+\beta,\alpha} f(x) \tag{3.3.12}$$

$$\text{(ii)} \quad I_{\eta,\alpha} I_{\eta+\alpha,\beta} = I_{\eta,\alpha+\beta} = I_{\eta+\alpha,\beta,} I_{\eta,\alpha} \tag{3.3.13}$$

$$\text{(iii)} \quad I^{-1}_{\eta,\alpha} = I_{\eta+\alpha,-\alpha}. \tag{3.3.14}$$

Remark 3.3.2. The results of Exercise 3.3.4 also hold for the operator, defined by

$$\mathbb{K}_{\eta,\alpha} f(x) = \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_x^\infty (t^2 - x^2)^{\alpha-1} t^{-2\alpha-2\eta+1} f(t) \mathrm{d}t, \tag{3.3.15}$$

where $\Re(\alpha) > 0$.

Remark 3.3.3. Operators more general than the operators defined by (3.3.11) and (3.3.15) are recently defined by Galué et al [Integral Transform & Spec. Funct. Vol. 9 (2000), No. 3, pp. 185-196] in the form

$$_aI_x^{\eta,\alpha} f(x) = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} t^\eta f(t) \mathrm{d}t, \tag{3.3.16}$$

where $\Re(\alpha) > 0$.

3.4. Generalized Kober Operators

Notation 3.4.1. $\mathbb{I}[\alpha,\beta,\gamma : m,\mu,\eta,a : f(x)], \mathbb{I}[f(x)]$

Notation 3.4.2. $\mathbb{I}[\alpha,\beta,\gamma : m,\mu,\delta,a : f(x)], \mathbb{I}[f(x)]$

Notation 3.4.3. $\mathbb{R}[f(x)], \mathbb{R}\left[{}^{\alpha,\beta,\gamma;}_{\sigma,\rho,a;} : f(x)\right]$

Notation 3.4.4. $\mathbb{K}[f(x)], \mathbb{K}\left[{}^{\alpha,\beta,\gamma;}_{\delta,\rho,a;} : f(x)\right]$

Notation 3.4.5. $I_{0,x}^{\alpha,\beta,\eta;} f(x)$ (Saigo, 1978)

Notation 3.4.6. $J_{x,\alpha}^{\alpha,\beta,\eta;} f(x)$ (Saigo, 1978)

Definition 3.4.1.

$$\mathbb{I}[f(x)] = \mathbb{I}[\alpha, \beta, \gamma : m, \mu, \eta, a : f(x)]$$
$$= \frac{\mu x^{-\eta-1}}{\Gamma(1-\alpha)} \int_0^x {}_2F_1(\alpha, \beta + m, \gamma; \frac{at^\mu}{x^\mu}) t^\eta f(t) \mathrm{d}t, \tag{3.4.1}$$

where ${}_2F_1(\cdot)$ is the Gauss hypergeometric function.

Definition 3.4.2.

$$\mathbb{I}[f(x)] = \mathbb{I}[\alpha, \beta, \gamma : m, \mu, \delta, a : f(x)]$$
$$= \frac{\mu x^\delta}{\Gamma(1-\alpha)} \int_x^\infty {}_2F_1(\alpha, \beta + m; \gamma; \frac{ax^\mu}{t^\mu}) t^{-\delta-1} f(t) \mathrm{d}t. \tag{3.4.2}$$

Operators defined by (3.4.1) and (3.4.2) exist under the following conditions:

(i) $1 \le p,\ \ q < \infty,\ \ p^{-1} + q^{-1} = 1,\ \ |\arg(1-a)| < \pi$
(ii) $\Re(1-\alpha) > m, \Re(\eta) > -\frac{1}{q}, \Re(\delta) > -\frac{1}{p}, \Re(\gamma - \alpha - \beta - m) > -1, m \in \mathbb{N}_0;\ \ \gamma \neq 0, -1, -2, \cdots$
(iii) $f \in L_p(0, \infty)$

Equations (3.4.1) and (3.4.2) are introduced by Kalla and Saxena (1969).

For $\gamma = \beta$, (3.4.1) and (3.4.2) reduce to generalized Kober operators, given by Saxena (1967).

Definition 3.4.3.

$$\mathbb{R}[f(x)] = \mathbb{R}\begin{bmatrix} \alpha,\beta,\gamma; \\ \sigma,\rho,a; \end{bmatrix} f(x)$$
$$= \frac{x^{-\sigma-\rho}}{\Gamma(\rho)} \int_0^x t^\sigma (x-t)^{\rho-1} {}_2F_1[\alpha, \beta; \gamma; a(1 - \frac{t}{x})] f(t) \mathrm{d}t. \tag{3.4.3}$$

Definition 3.4.4.

$$\mathbb{K}[f(x)] = \mathbb{K}\begin{bmatrix} \alpha,\beta,\gamma; \\ \delta,\rho,a; \end{bmatrix} f(x)$$
$$= \frac{x^\delta}{\Gamma(\rho)} \int_x^\infty t^{-\delta-\rho} (t-x)^{\rho-1} {}_2F_1[\alpha, \beta; \gamma; a(1 - \frac{x}{t})] f(t) \mathrm{d}t. \tag{3.4.4}$$

The conditions of validity of the operators (3.4.3) and (3.4.4) are given below:

(i) $p \geq 1, \quad q < \infty, \quad p^{-1} + q^{-1} = 1, \quad |\arg(1-a)| < \pi.$
(ii) $\Re(\sigma) > -\frac{1}{q}, \Re(\delta) > -\frac{1}{p}, \Re(\rho) > 0.$
(iii) $\gamma \neq 0, -1, -2, \cdots; \Re(\gamma - \alpha - \beta) > 0.$
(iv) $f \in L_p(0, \infty).$

The operators defined by (3.4.3) and (3.4.4) are given by Saxena and Kumbhat (1973). When a is replaced by $\frac{a}{\alpha}$ and α tends to infinity, the operators defined by (3.4.3) and (3.4.4) reduce to the following operators associated with confluent hypergeometric functions.

Definition 3.4.5.

$$\mathbb{R}\left[\begin{smallmatrix}\beta,\gamma;\\ \sigma,\rho,a;\end{smallmatrix} f(x)\right] = \lim_{\alpha\to\infty} \mathbb{R}\left[\begin{smallmatrix}\alpha,\beta,\gamma;\\ \sigma,\rho,\frac{a}{\alpha};\end{smallmatrix} f(x)\right]$$
$$= \frac{x^{-\sigma-\rho}}{\Gamma(\rho)} \int_0^x \Phi[\beta, \gamma; a(1-\frac{t}{x})] t^{\sigma}(x-t)^{\rho-1} f(t)\mathrm{d}t. \tag{3.4.5}$$

Definition 3.4.6.

$$\mathbb{K}\left[\begin{smallmatrix}\beta,\gamma;\\ \sigma,\rho,a;\end{smallmatrix} f(x)\right] = \lim_{\alpha\to\infty} \mathbb{K}\left[\begin{smallmatrix}\alpha,\beta,\gamma;\\ \delta,\rho,\frac{a}{\alpha};\end{smallmatrix} f(x)\right]$$
$$= \frac{x^{\delta}}{\Gamma(\rho)} \int_x^\infty \Phi[\beta, \gamma; a(1-\frac{x}{t})] t^{-\delta-\rho}(t-x)^{\rho-1} f(t)\mathrm{d}t, \tag{3.4.6}$$

where $\Re(\rho) > 0, \Re(\delta) > 0.$

Remark 3.4.1. Many interesting and useful properties of the operators defined by (3.4.3) and (3.4.4) are investigated by Saxena and Kumbhat (1975), which deal with relations of these operators with well-known integral transforms, such as Laplace, Mellin and Hankel transforms. Equation (3.4.3) was first considered by Love (1967).

Remark 3.4.2. In the special case, when α is replaced by $\alpha + \beta, \gamma$ by α, σ by zero, ρ by α and β by $-\eta$, then (3.4.3) reduces to the operator (3.4.7) considered by Saigo (1978). Similarly, (3.4.4) reduces to another operator (3.4.9) introduced by Saigo (1978).

Definition 3.4.7. Let $\alpha, \beta, \eta \in \mathbb{C}$, and let $x \in \mathbb{R}_+$ the fractional integral ($\Re(\alpha) > 0$) and the fractional derivative ($\Re(\alpha) < 0$) of the first kind of a function $f(x)$ on $\mathbb{R}_+$ are defined by Saigo (1978) in the form

$$\begin{aligned} I_{0,x}^{\alpha,\beta,\eta} f(x) &= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \\ &\quad \times {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}) f(t) \mathrm{d}t, \ \Re(\alpha) > 0 \qquad (3.4.7) \\ &= \frac{\mathrm{d}^n}{\mathrm{d}x^n} I_{0,x}^{\alpha+n,\beta-n,\eta-n} f(x), \ 0 < \Re(\alpha) + n \leq 1, \ (n \in \mathbb{N}_0). \qquad (3.4.8) \end{aligned}$$

Definition 3.4.8. The fractional integral ($\Re(\alpha) > 0$) and fractional derivative ($\Re(\alpha) < 0$) of the second kind of a function $f(x)$ on $\mathbb{R}_+$ are given by Saigo (1978) in the form

$$\begin{aligned} J_{x,\infty}^{\alpha,\beta,\eta} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} \\ &\quad \times {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}) f(t) \mathrm{d}t, \ \Re(\alpha) > 0 \qquad (3.4.9) \\ &= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} J_{x,\infty}^{\alpha+n,\beta-n,\eta} f(x), \ 0 < \Re(\alpha) + n \leq 1, \ (n \in \mathbb{N}_0). \qquad (3.4.10) \end{aligned}$$

Example 3.4.1. Find the value of

$$I_{0,x}^{\alpha,\beta,\eta} \left\{ x^{\sigma-1} \ {}_2F_1(a, b; c; -a' x) \right\}.$$

Solution 3.4.1. We have

$$\begin{aligned} K &= I_{0,x}^{\alpha,\beta,\eta} \left\{ x^{\sigma-1} \ {}_2F_1(a, b; c; -ax) \right\} \\ &= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (-1)^r (a')^r}{(c)_r r!} I_{0,x}^{\alpha,\beta,\eta} x^{r+\sigma-1}. \end{aligned}$$

Applying the result of Exercise 3.4.1, we obtain

$$K = x^{\sigma-\beta-1} \sum_{r=0}^{\infty}(-1)^r \frac{(a)_r(b)_r \ \Gamma(\sigma+r)\Gamma(\sigma-\beta+\eta+r)(a')^r}{(c)_r r! \ \Gamma(\sigma-\beta+r)\Gamma(\alpha+\eta+\sigma+r)} x^r$$
$$= x^{\sigma-\beta-1} \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)}$$
$$\times \ {}_4F_3(a,b,\sigma,\sigma+\eta-\beta; c,\sigma-\beta,\sigma+\alpha+\eta; -a'x),$$

where $\Re(\alpha) > 0, \Re(\sigma) > 0, \Re(\sigma+\eta-\beta) > 0, c \neq 0,-1,-2,\cdots; \ |a'x| < 1$.

Example 3.4.2. Find the value of

$$J_{x,\infty}^{\alpha,\beta,\eta}(x^{\lambda} \ {}_2F_1(a,b;c; \ \frac{a'}{x})).$$

Solution 3.4.2. Following a similar procedure and using the result of Exercise 3.4.3, it gives

$$J_{x,\infty}^{\alpha,\beta,\eta}(x^{\lambda} \ {}_2F_1(a,b;c; \ \frac{a'}{x})) = \frac{\Gamma(\beta-\lambda)\Gamma(\eta-\lambda)}{\Gamma(-\lambda)\Gamma(\alpha+\beta+\eta-\lambda)} x^{\lambda-\beta}$$
$$\times \ {}_4F_3(a,b,\beta-\lambda,\eta-\lambda; c,-\lambda,\alpha+\beta+\eta-\lambda; \frac{a'}{x}),$$

where $\Re(\alpha) > 0, \Re(\beta-\lambda) > 0, \Re(\eta-\lambda) > 0, x > 0, c \neq 0,-1,-2,\cdots; \ |x| > |a'|$.

Remark 3.4.3. Special cases of the operators $I_{0,x}^{\alpha,\beta,\eta}$ and $J_{x,\infty}^{\alpha,\beta,\eta}$ are the operators of Riemann -Liouville:

$$I_{0,x}^{\alpha,-\alpha,\eta} f(x) = {}_0D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \ (\Re(\alpha) > 0) \tag{3.4.11}$$

the Weyl:

$$J_{x,\infty}^{\alpha,-\alpha,\eta} f(x) = {}_xW_{\infty}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t)\mathrm{d}t, \ (\Re(\alpha) > 0) \tag{3.4.12}$$

and the Erdélyi - Kober operators:

$$I_{0,x}^{\alpha,0,\eta}\ f(x) = E_{0,x}^{\alpha,\eta}\ f(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^{\eta} f(t)\mathrm{d}t,\ (\Re(\alpha) > 0) \tag{3.4.13}$$

and

$$J_{x,\infty}^{\alpha,0,\eta}\ f(x) = K_{x,\infty}^{\alpha,\eta}\ f(x) = \frac{x^{\eta}}{\Gamma(\alpha)}\int_x^{\infty} (t-x)^{\alpha-1}t^{-\alpha-\eta} f(t)\mathrm{d}t,\ (\Re(\alpha) > 0) \tag{3.4.14}$$

Example 3.4.3. Prove the following theorem.

If $\Re(\alpha) > 0$ and $\Re(s) < 1 + \min[0, \Re(\eta-\beta)]$, then the following formula holds for $f(x) \in L_p(0,\infty)$ with $1 \le p \le 2$ or $f(x) \in M_p(0,\infty)$ with $p > 2$:

$$m\left\{x^{\beta} I_{0,x}^{\alpha,\beta,\eta} f\right\} = \frac{\Gamma(1-s)\Gamma(\eta-\beta+1-s)}{\Gamma(1-s-\beta)\Gamma(\alpha+\eta+1-s)} m\{f(x)\}. \tag{3.4.15}$$

Solution 3.4.3. Use the integral

$$\int_x^{\infty} u^{-\sigma-\gamma}(u-x)^{\gamma-1}\ {}_2F_1\ (\alpha,\beta;\gamma;1-\frac{x}{u})\mathrm{d}u = \frac{\Gamma(\gamma)\Gamma(\sigma)\Gamma(\gamma+\sigma-\alpha-\beta)}{\Gamma(\gamma+\sigma-\alpha)\Gamma(\gamma+\sigma-\beta)}, \tag{3.4.16}$$

where $\Re(\gamma) > 0,\ \Re(\sigma) > 0,\ \Re(\gamma+\sigma-\alpha-\beta) > 0.$

Exercises 3.4.

3.4.1. Prove that

$$I_{0,x}^{\alpha,\beta,\eta}\ x^{\lambda} = \frac{\Gamma(1+\lambda)\Gamma(1+\lambda+\eta-\beta)}{\Gamma(1+\lambda-\beta)\Gamma(1+\lambda+\alpha+\eta)} x^{\lambda-\beta}, \tag{3.4.17}$$

and give the conditions of validity.

3.4.2. Find the Mellin transform of $x^{\beta}\ J_{x,\infty}^{\alpha,\beta,\eta}\ f(x)$, giving conditions of its validity.

3.4.3. Prove that

$$J_{x,\infty}^{\alpha,\beta,\eta}\ x^{\lambda} = \frac{\Gamma(\beta-\lambda)\Gamma(\eta-\lambda)}{\Gamma(-\lambda)\Gamma(\alpha+\beta+\eta-\lambda)} x^{\lambda-\beta} \tag{3.4.18}$$

and give the conditions of validity.

3.4.4. Prove that

$$I_{0,x}^{\alpha,\beta,\eta}\,(x^k e^{-\lambda x}) = \frac{\Gamma(k+1)\Gamma(\eta+k-\beta+1)}{\Gamma(k-\beta+1)\Gamma(\alpha+\eta+k+1)} x^{k-\beta} \times {}_2F_2(k+1, \eta+k-\beta+1;\ \ k-\beta+1, \alpha+\eta+k+1; -\lambda x), \quad (3.4.19)$$

and give the conditions of validity.

3.4.5. Prove that

$$J_{x,\infty}^{\alpha,\beta,\eta}\, e^{-sx} = s^{\eta} x^{\eta-\beta} \frac{\Gamma(\beta-\eta)}{\Gamma(\alpha+\beta)} \Phi(1-\alpha-\beta, 1+\eta-\beta; -sx) + s^{\beta} \frac{\Gamma(\eta-\beta)}{\Gamma(\alpha+\eta)} \Phi(1-\alpha-\eta, 1+\beta-\eta; -sx), \quad (3.4.20)$$

and give the conditions of its validity. Deduce the results for $L[\ {}_xW_{\infty}^{\alpha}\ f](s)$ and $L[\ K_{x,\infty}^{\alpha,\eta}\ f](s)$.

3.4.6. Prove that [Saxena and Nishimoto (2002)]

$$I_{0,x}^{\alpha,\beta,\eta}\,[x^{\sigma-1}(a+bx)^c] = a^c \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)} x^{\sigma-\beta-1} \times {}_3F_2(\sigma, \sigma+\eta-\beta, -c;\ \sigma-\beta, \sigma+\alpha+\eta; -\frac{bx}{a}), \quad (3.4.21)$$

where $\Re(\sigma) > \max[0, \Re(\beta-\eta)], |\frac{bx}{a}| < 1$.

3.4.7. Evaluate

$$I_{0,x}^{\alpha,\beta,\eta}\ \left\{x^{\sigma-1} H_{p,q}^{m,n}\left[ax^{\lambda}\ \Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\},\ \lambda > 0, \quad (3.4.22)$$

and give the conditions of its validity.

3.4.8. Evaluate

$$J_{x,\infty}^{\alpha,\beta,\eta}\ \left\{x^{\sigma-1} H_{p,q}^{m,n}\left[ax^{-\lambda}\ \Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\},\ \lambda > 0, \quad (3.4.23)$$

and give the conditions of its validity.

3.4.9. Establish the following property of Saigo operators called "Integration by parts".

$$\int_0^\infty f(x)\left(I_{0,x}^{\alpha,\beta,\eta} g\right)(x)\mathrm{d}x = \int_0^\infty g(x)\left(J_{x,\infty}^{\alpha,\beta,\eta} f\right)(x)\mathrm{d}x.$$

3.4.10. From Exercise 3.4.6, deduce the formula for

$$I_{0,x}^{\alpha,-\alpha,\eta}(a+bx)^c, \tag{3.4.24}$$

given by B. Ross (1993).

3.4.11. Prove that

$$R_{0,x}^{\alpha} x^k = \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)} x^{k+\alpha}, \tag{3.4.25}$$

where $\Re(\alpha) > 0,\ \Re(k) > -1,$

3.4.12. Prove that

$$W_{x,\infty}^{\alpha} x^k = \frac{\Gamma(-\alpha-k)}{\Gamma(-k)} x^{k+\alpha}, \tag{3.4.26}$$

where $\Re(\alpha) > 0,\ \Re(k) < -\Re(\alpha).$

3.4.13. Show that

$$J_{x,\infty}^{\alpha,\beta,\eta}\left(x^{\lambda}\mathrm{e}^{-px}\right) = x^{\lambda-\beta} G_{2,3}^{3,0}\left[px\Big|_{0,\beta-\lambda,\eta-\lambda}^{-\lambda,\alpha+\beta+\eta-\lambda}\right], \tag{3.4.27}$$

where $G_{2,3}^{3,0}(\cdot)$ is the Meijer's G-function, $\Re(px) > 0,\ \Re(\alpha) > 0.$

Hint: Use the integral

$$\mathrm{e}^{-px} = \frac{1}{2\pi i}\int_L \Gamma(-s)(px)^s \mathrm{d}s. \tag{3.4.28}$$

3.4.14. Evaluate

$$I_{0,x}^{\alpha,\beta,\eta} x^{\sigma-1} H_{p,q}^{m,n}\left[ax^{-\lambda}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right],\quad \lambda > 0, \tag{3.4.29}$$

giving the conditions of its validity.

3.4.15. Evaluate

$$J_{x,\infty}^{\alpha,\beta,\eta} x^{\sigma-1} H_{p,q}^{m,n}\left[ax^{\lambda}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right], \quad \lambda > 0 \tag{3.4.30}$$

and give the conditions of validity of the result.

3.4.16. With the help of the following chain rules for Saigo operators (Saigo, 1985)

$$I_{0,x}^{\alpha,\beta,\eta} I_{0,x}^{\gamma,\delta,\alpha+\eta} f = I_{0,x}^{\alpha+\gamma,\beta+\delta,\eta} f, \tag{3.4.31}$$

and

$$J_{x,\infty}^{\alpha,\beta,\eta} J_{x,\infty}^{\gamma,\delta,\alpha+\eta} f = J_{x,\infty}^{\alpha+\gamma,\beta+\delta,\eta} f, \tag{3.4.32}$$

derive the inverses

$$(I_{0,x}^{\alpha,\beta,\eta})^{-1} = I_{0,x}^{-\alpha,-\beta,\alpha+\eta}. \tag{3.4.33}$$

and

$$(J_{x,\infty}^{\alpha,\beta,\eta})^{-1} = J_{x,\infty}^{-\alpha,-\beta,\alpha+\eta}. \tag{3.4.34}$$

3.5. Compositions of Riemann-Liouville Fractional Calculus Operators and Generalized Mittag-Leffler Functions

In this section, composition relations between Riemann-Liouville fractional calculus operators and generalized Mittag-Leffler functions are derived. These relations may be useful in the solution of fractional differintegral equations. For details, one can refer to the work of Saxena and Saigo (2005).

3.5.1. Composition relations between R-L operators and $E^{\delta}_{\beta,\gamma}(z)$

Notation 3.5.1. $E_\alpha(x)$: Mittag-Leffler function.

Notation 3.5.2. $E_{\alpha,\beta}(x)$: Generalized Mittag-Leffler function.

Notation 3.5.3. $I^{\alpha}_{0+} f$: Riemann-Liouville left-sided integral.

Notation 3.5.4. $I^{\alpha}_{-} f$: Riemann-Liouville right-sided integral.

Notation 3.5.5. $D^{\alpha}_{0+} f$: Riemann-Liouville left-sided derivative.

Notation 3.5.6. $D^{\alpha}_{-} f$: Riemann-Liouville right-sided derivative.

Notation 3.5.7. $E^{\delta}_{\beta,\gamma}(z)$: Generalized Mittag-Leffler function (Prabhakar, 1971).

Definition 3.5.1.

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad (\alpha \in C, \Re(\alpha) > 0). \tag{3.5.1}$$

Definition 3.5.2.

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha, \beta \in C, \Re(\alpha) > 0, \Re(\beta) > 0). \tag{3.5.2}$$

Definition 3.5.3.

$$(I^{\alpha}_{0+} f)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} \mathrm{d}t, \quad \Re(\alpha) > 0. \tag{3.5.3}$$

Definition 3.5.4.

$$(I^{\alpha}_{-} f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \frac{f(t)}{(t-x)^{1-\alpha}} \mathrm{d}t, \quad \Re(\alpha) > 0. \tag{3.5.4}$$

Definition 3.5.5.

$$(D^{\alpha}_{0+} f)(x) : = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1} \left(I^{1-\{\alpha\}}_{0+}\right)(x); \quad \Re(\alpha) > 0 \tag{3.5.5}$$

$$= \frac{1}{\Gamma(1-\{\alpha\})} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1} \int_0^x \frac{f(t)}{(x-t)^{\{\alpha\}}} \mathrm{d}t, \quad \Re(\alpha) > 0. \tag{3.5.6}$$

Definition 3.5.6.

$$(D_-^\alpha f)(x) := \left(\frac{d}{dx}\right)^{[\alpha]+1} (I_-^{1-\{\alpha\}} f)(x), \quad \Re(\alpha) > 0 \tag{3.5.7}$$

$$= \frac{1}{\Gamma(1-\{\alpha\})}\left(-\frac{d}{dx}\right)^{[\alpha]+1} \int_x^\infty \frac{f(t)}{(t-x)^{\{\alpha\}}}dt, \quad \Re(\alpha) > 0. \tag{3.5.8}$$

Remark 3.5.1. Here $[\alpha]$ means the maximal integer not exceeding α and $\{\alpha\}$ is the fractional part of α.

Definition 3.5.7.

$$E_{\beta,\gamma}^\delta(z) := \sum_{k=0}^\infty \frac{(\delta)_k z^k}{\Gamma(\beta k+\gamma)k!}, \quad (\beta, \gamma, \delta \in C; \Re(\gamma) > 0, \Re(\beta) > 0). \tag{3.5.9}$$

For $\delta = 1$, (3.5.9) reduces to (3.5.2).

Theorem 3.5.1. *Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\alpha \in \mathbb{R}$. Let I_{0+}^α be the left-sided operator of Riemann-Liouville fractional integral (3.5.3). Then there holds the formula*

$$(I_{0+}^\alpha[t^{\gamma-1}E_{\beta,\gamma}^\delta(at^\beta)])(x) = x^{\alpha+\gamma-1}E_{\beta,\alpha+\gamma}^\delta(ax^\beta). \tag{3.5.10}$$

Proof 3.5.1. By virtue of (3.5.3) and (3.5.9), we have

$$K \equiv (I_{0+}^\alpha[t^{\gamma-1}E_{\beta,\gamma}^\delta(at^\beta)])(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\sum_{n=0}^\infty \frac{(\delta)_n a^n t^{n\beta+\gamma-1}}{\Gamma(\beta n+\gamma)n!}dt.$$

Interchanging the order of integration and summation and evaluating the inner integral by means of beta-function formula, it gives

$$K \equiv x^{\alpha+\gamma-1}\sum_{n=0}^\infty \frac{(\delta)_n (ax^\beta)^n}{\Gamma(\alpha+\beta n+\gamma)(n)!} = x^{\alpha+\gamma-1}E_{\beta,\alpha+\gamma}^\delta(ax^\beta).$$

This completes the proof of Theorem 3.5.1.

Corollary 3.5.1. For $\alpha > 0, \beta > 0, \gamma > 0$ and $\alpha \in \mathbb{R}$, there holds the formula

$$(I_{0+}^\alpha[t^{\gamma-1}E_{\beta,\gamma}(at^\beta)])(x) = x^{\alpha+\gamma-1}E_{\beta,\alpha+\gamma}(ax^\beta). \tag{3.5.11}$$

Remark 3.5.2. For $\beta = \alpha$, (3.5.11) reduces to

$$(I^{\alpha}_{0+}[t^{\gamma-1}E_{\alpha,\gamma}(at^{\beta})])(x) = \frac{x^{\gamma-1}}{a}[E_{\alpha,\gamma}(ax^{\alpha}) - \frac{1}{\Gamma(\gamma)}], (a \neq 0) \tag{3.5.12}$$

by virtue of the identity

$$E_{\alpha,\gamma}(x) = \frac{1}{\Gamma(\gamma)} + xE_{\alpha,\alpha+\gamma}(x), (a \neq 0). \tag{3.5.13}$$

Theorem 3.5.2. *Let $\alpha > 0, \beta > 0, \gamma > 0$ and $\alpha \in \mathbb{R}$, $(a \neq 0)$ and let I^{α}_{0+} be the left-sided operator of Riemann-Liouville fractional integral (3.5.3). Then there holds the formula*

$$(I^{\alpha}_{0+}[t^{\gamma-1}E^{\delta}_{\beta,\gamma}(at^{\beta})])(x) = \frac{1}{a}x^{\alpha+\gamma-\beta-1}[E^{\delta}_{\beta,\alpha+\gamma-\beta}(ax^{\beta}) - E^{\delta-1}_{\beta,\alpha+\gamma-\beta}(ax^{\beta})]. \tag{3.5.14}$$

Proof. Use Theorem 3.5.1.

The following two theorems can be established in the same way.

Theorem 3.5.3. *Let $\alpha > 0, \beta > 0, \gamma > 0$ and $\alpha \in \mathbb{R}$ and let I^{α}_{-} be the right-sided operator of Riemann-Liouville fractional integral (3.5.4). Then we arrive at the following result:*

$$(I^{\alpha}_{-}[t^{-\alpha-\gamma}E^{\delta}_{\beta,\gamma}(at^{-\beta})])(x) = x^{-\gamma}[E^{\delta}_{\beta,\alpha+\gamma}(ax^{-\beta})] \tag{3.5.15}$$

Corollary 3.5.2. For $\alpha > 0, \beta > 0, \gamma > 0$ and $\alpha \in \mathbb{R}$, there holds the formulas:

$$(I^{\alpha}_{-}[t^{-\alpha-\gamma}E_{\beta,\gamma}(at^{-\beta})])(x) = x^{-\gamma}[E_{\beta,\alpha+\gamma}(ax^{-\beta})] \tag{3.5.16}$$

and

$$(I^{\alpha}_{-}t^{-\alpha-1}E_{\beta}(at^{-\beta}))(x) = x^{-1}[E_{\beta,\alpha+1}(ax^{-\beta})]. \tag{3.5.17}$$

Theorem 3.5.4. *Let $\alpha > 0, \beta > 0, \gamma > 0, \alpha \in \mathbb{R}$, $(a \neq 0), \alpha + \gamma > \beta$ and let I^{α}_{-} be the right-sided operator of Riemann-Liouville fractional integral (3.5.4). Then there holds the formula*

$$(I_-^\alpha[t^{-\alpha-\gamma}E_{\beta,\gamma}^\delta(at^{-\beta})])(x) = \frac{1}{a}x^{\beta-\gamma}[E_{\beta,\alpha+\gamma-\beta}^\delta(ax^{-\beta}) - E_{\beta,\alpha+\gamma-\beta}^{\delta-1}(ax^{-\beta})]. \quad (3.5.18)$$

Corollary 3.5.3. For $\alpha > 0, \beta > 0, \gamma > 0$ with $\alpha + \gamma > \beta$ and for $\alpha \in \mathbb{R}$, $(a \neq 0)$, there holds the formula

$$(I_-^\alpha[t^{-\alpha-\gamma}E_{\beta,\gamma}(at^{-\beta})])(x) = \frac{1}{a}x^{\beta-\gamma}\left[E_{\beta,\alpha+\gamma-\beta}(ax^{-\beta}) - \frac{1}{\Gamma(\alpha+\gamma-\beta)}\right]. \quad (3.5.19)$$

Remark 3.5.3. (Kilbas and Saigo, (1998))

$$(I_-^\alpha[t^{-\alpha-\gamma}E_{\alpha,\gamma}(at^{-\alpha})])(x) = \frac{x^{\alpha-\gamma}}{a}[E_{\alpha,\gamma}(ax^{-\alpha}) - \frac{1}{\Gamma(\gamma)}], \; (a \neq 0) \quad (3.5.20)$$

$$(I_-^\alpha[t^{-\alpha-1}E_\alpha(at^{-\alpha})])(x) = \frac{x^{\alpha-1}}{a}\left[E_\alpha(ax^{-\alpha}) - 1\right], \; (a \neq 0). \quad (3.5.21)$$

Theorem 3.5.5. *Let $\alpha > 0, \beta > 0, \gamma > 0, \gamma > \alpha, \alpha \in \mathbb{R}$ and let D_{0+}^α be the left-sided operator of Riemann -Liouville fractional derivative (3.5.6). Then there holds the formula.*

$$(D_{0+}^\alpha[t^{\gamma-1}E_{\beta,\gamma}^\delta(at^\beta)])(x) = x^{\gamma-\alpha-1}E_{\beta,\gamma-\alpha}^\delta(ax^\beta). \quad (3.5.22)$$

Proof 3.5.2. By virtue of (3.5.9) and (3.5.6), we have

$$\begin{aligned}
K \equiv (D_{0+}^\alpha[t^{\gamma-1}E_{\beta,\gamma}^\delta(at^\beta)])(x) &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}\left(I_{0+}^{1-\{\alpha\}}\left[t^{\gamma-1}E_{\beta,\gamma}^\delta(at^\beta)\right]\right)(x) \\
&= \sum_{n=0}^\infty \frac{a^n(\delta)_n}{\Gamma(\gamma+n\beta)\Gamma(1-\{a\})n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}\int_0^x t^{n\beta+\gamma-1}(x-t)^{-\{\alpha\}}\mathrm{d}t \\
&= \sum_{n=0}^\infty \frac{a^n(\delta)_n}{\Gamma(\gamma+n\beta+1-\{\alpha\})n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1} x^{n\beta+\gamma-\{\alpha\}} \\
&= \sum_{n=0}^\infty \frac{a^n(\delta)_n x^{\gamma+n\beta-\alpha-1}}{\Gamma(n\beta+\gamma-\alpha)n!} = x^{\gamma-\alpha-1}E_{\beta,\gamma-\alpha}^\delta(ax^\beta),
\end{aligned}$$

which proves the theorem.

By using a similar procedure, we arrive at the following theorem.

Theorem 3.5.6. *Let $\alpha > 0, \gamma > \beta > 0, \alpha \in \mathbb{R}$, $(a \neq 0)$, $\gamma > \alpha + \beta$ and let D^{α}_{0+} be the left-sided operator of Riemann-Liouville fractional derivative (3.5.6). Then there holds the formula*

$$\left(D^{\alpha}_{0+}[t^{\gamma-1}E^{\delta}_{\beta,\gamma}(at^{\beta})]\right)(x) = \frac{1}{a}x^{\gamma-\alpha-\beta-1}\left[E^{\delta}_{\beta,\gamma-\alpha-\beta}(ax^{\beta}) - E^{\delta-1}_{\beta,\gamma-\alpha-\beta}(ax^{\beta})\right]. \quad (3.5.23)$$

Corollary 3.5.4. Let $\alpha > 0, \gamma > \beta > 0, \alpha \in \mathbb{R}$, $(a \neq 0)$, $\gamma > \alpha + \beta$, then there holds the formula.

$$\left(D^{\alpha}_{0+}[t^{\gamma-1}E_{\beta,\gamma}(at^{\beta})]\right)(x) = \frac{1}{a}x^{\gamma-\alpha-\beta-1}\left[E_{\beta,\gamma-\alpha-\beta}(ax^{\beta}) - \frac{1}{\Gamma(\gamma-\alpha-\beta)}\right]. \quad (3.5.24)$$

Theorem 3.5.7. *Let $\alpha > 0, \gamma > 0, \gamma - \alpha > 0$ with $\gamma - \alpha + \{\alpha\} > 1, \alpha \in \mathbb{R}$, and let D^{α}_{-} be the right-sided operator of Riemann-Liouville fractional derivative (3.5.8). Then there holds the formula.*

$$\left(D^{\alpha}_{-}[t^{\alpha-\gamma}E^{\delta}_{\beta,\gamma}(at^{-\beta})]\right)(x) = x^{-\gamma}E^{\delta}_{\beta,\gamma-\alpha}(ax^{-\beta}). \quad (3.5.25)$$

Theorem 3.5.8. *Let $\alpha > 0, \beta > 0$ with $\gamma - \{\alpha\} > 1$, $\alpha \in \mathbb{R}$, $\gamma > \alpha+\beta$, $(a \neq 0)$ and let D^{α}_{-} be the right-sided operator of Riemann-Liouville fractional derivative (3.5.8). Then there holds the formula*

$$\left(D^{\alpha}_{-}[t^{\alpha-\gamma}E^{\delta}_{\beta,\gamma}(at^{-\beta})]\right)(x) = \frac{x^{\beta-\gamma}}{a}\left[E^{\delta}_{\beta,\gamma-\alpha-\beta}(ax^{-\beta}) - E^{\delta-1}_{\beta,\gamma-\alpha-\beta}(ax^{-\beta})\right]. \quad (3.5.26)$$

Exercises 3.5.

3.5.1. Show that

$$ax^{\beta}E^{\delta}_{\beta,\gamma}(ax^{\beta}) = E^{\delta}_{\beta,\gamma-\beta}(ax^{\beta}) - E^{\delta-1}_{\beta,\gamma-\beta}(ax^{\beta}), (a \neq 0) \quad (3.5.27)$$

3.5.2. Show that

$$\left(I_{0+}^{\alpha}[t^{\gamma-1}E_{\alpha,\gamma}(at^{\alpha})]\right)(x) = \frac{x^{\gamma-1}}{a}\left[E_{\alpha,\gamma}(ax^{\alpha}) - \frac{1}{\Gamma(\gamma)}\right],\ (a \neq 0). \tag{3.5.28}$$

3.5.3. Prove Theorem 3.5.3.

3.5.4. Prove Theorem 3.5.4.

3.5.5. Prove Theorem 3.5.6.

3.5.6. Prove Theorem 3.5.7.

3.5.7. Prove Theorem 3.5.8.

3.5.8. Prove that

$$\left(I_{0+}^{\alpha} t^{\omega} H_{p,q}^{m,n}\left[t^{\sigma}\Big|_{(bq,Bq)}^{(a_p,A_p)}\right]\right)(x) = x^{\omega+\alpha} H_{p+1,q+1}^{m,n+1}\left[x^{\sigma}\Big|_{(bq,Bq),(-\omega-\alpha,\sigma)}^{(-\omega,\sigma),(a_p,A_p)}\right], \tag{3.5.29}$$

giving conditions of validity.

3.5.9. Evaluate

$$\left(I_{-}^{\alpha} t^{\omega} H_{p,q}^{m,n}\left[t^{\sigma}\Big|_{(bq,Bq)}^{(a_p,A_p)}\right]\right)(x), \tag{3.5.30}$$

and give the conditions of validity.

3.6. Fractional Differential Equations

Differential equations contain integer order derivatives, whereas fractional differential equations involve fractional derivatives, like $\frac{d^{\alpha}}{dx^{\alpha}}$, which are defined for $\alpha > 0$. Here α is not necessarily an integer and can be rational, irrational or even complex-valued. Today, fractional calculus models find applications in physical, biological, engineering, biomedical and earth sciences. Most of the problems discussed involve relaxation and diffusion models in the so called complex or disordered systems. Thus, it gives rise to the generalization of initial value problems involving ordinary differential equations to generalized fractional-order differential equations and Cauchy problems involving partial differential equations to fractional reaction, fractional diffusion and fractional reaction-diffusion equations. Fractional calculus plays a dominant role in the solution of all these physical problems.

3.6.1. Fractional relaxation

In order to formulate a relaxation process, we require a physical law, say the relaxation equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) + \frac{1}{c} f(t) = 0, t > 0, c > 0, \tag{3.6.1}$$

to be solved for the initial value $f(t = 0) = f_0$. The unique solution of (3.6.1) is given by

$$f(t) = f_0 \ \mathrm{e}^{-\frac{t}{c}}, t \geq 0, c > 0. \tag{3.6.2}$$

Now the problem is as to how we can generalize the initial-value problem (3.6.1) into a fractional value problem with physical motivation.

If we incorporate the initial value f_0 into the integrated relaxation equation (3.6.1), we find that

$$f(t) - f_0 = -\frac{1}{c} \ {}_0D_t^{-1} \ f(t), \tag{3.6.3}$$

where ${}_0D_t^{-1}$ is the standard Riemann integral of $f(t)$.

On replacing $\frac{1}{c} \ {}_0D_t^{-1} \ f(t)$ by $\frac{1}{c^\alpha} \ {}_0D_t^{-\alpha} \ f(t)$, it yields the fractional integral equation

$$f(t) - f_0 = -\left(\frac{1}{c^\alpha}\right) {}_0D_t^{-\alpha} \ f(t), \alpha > 0 \tag{3.6.4}$$

with initial value

$$f_0 = f(t = 0).$$

Applying the Riemann-Liouville differential operator ${}_0D_t^\alpha$ from the left and making use of the formula (3.1.16), we arrive at

$${}_0D_t^\alpha f(t) = f_0 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = -c^{-\alpha} f(t), \ \alpha > 0, c > 0, \tag{3.6.5}$$

with initial condition $f_0 = f(t = 0)$.

Theorem 3.6.1. *The solution of the fractional differential equation (3.6.4) is given by*

$$f(t) = f_0 H_{1,2}^{1,1}\left[\left(\frac{t}{c}\right)^{\alpha}\Big|_{(0,1),(0,\alpha)}^{(0,1)}\right], \tag{3.6.6}$$

where $\alpha > 0, c > 0$.

Proof 3.6.1. If we apply the Laplace transform to equation (3.6.4), it gives

$$F(s) - f_0 s^{-1} = -\frac{1}{c^{\alpha}} s^{-\alpha} F(s), \tag{3.6.7}$$

where we have used the result (3.1.7) and $F(s)$ is the Laplace transform of $f(t)$. Solving for $F(s)$, we have

$$F(s) = L\{f(t)\} = f_0\left[\frac{s^{-1}}{1+(cs)^{-\alpha}}\right]. \tag{3.6.8}$$

Taking inverse Laplace transform, (3.6.8) gives

$$\begin{aligned} f(t) = L^{-1}\{F(s)\} &= f_0 L^{-1}\left[\frac{s^{-1}}{1+(cs)^{-\alpha}}\right] \\ &= f_0 L^{-1}\left[\sum_{k=0}^{\infty}(-1)^k c^{-\alpha k} s^{-\alpha k-1}\right] \\ &= f_0 \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{t}{c})^{\alpha k}}{\Gamma(\alpha k+1)} \\ &= f_0 E_{\alpha}\left[-(\frac{t}{c})^{\alpha}\right], \end{aligned} \tag{3.6.9}$$

where $E_{\alpha}(\cdot)$ is the Mittag-Leffler function. (3.6.9) can be written in terms of the H-function as

$$f(t) = f_0 H_{1,2}^{1,1}\left[\left(\frac{t}{c}\right)^{\alpha}\Big|_{(0,1),(0,\alpha)}^{(0,1)}\right], \tag{3.6.10}$$

where $c > 0, \alpha > 0$. This completes the proof of the Theorem 3.6.1.

Alternative form of the solution. By virtue of the identity

$$H_{p,q}^{m,n}\left[x^{\mu}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = \frac{1}{\mu} H_{p,q}^{m,n}\left[x\Big|_{(b_q,\frac{B_q}{\mu})}^{(a_p,\frac{A_p}{\mu})}\right],\ (\mu > 0) \tag{3.6.11}$$

the solution (3.6.10) can be written as

$$f(t) = \frac{f_0}{\alpha} H_{1,2}^{1,1}\left[\frac{t}{c}\Big|_{(0,\frac{1}{\alpha}),(0,1)}^{(0,\frac{1}{\alpha})}\right], \tag{3.6.12}$$

where $\alpha > 0, c > 0$.

Remark 3.6.1. In the limit as $\alpha \to 1$, one recovers the result (3.6.2)

$$f(t) = f_0 \exp(-\frac{t}{c}) = f_0 E_1(\frac{t}{c}). \tag{3.6.13}$$

Remark 3.6.2. In terms of Wright's function, the solution (3.6.10) can be expressed in the form

$$f(t) = f_0 \; {}_1\psi_1 \left[\begin{matrix}(1,1)\\(1,\alpha)\end{matrix}\Big| - (\frac{t}{c})^{\alpha}\right], \tag{3.6.14}$$

where $\alpha > 0, c > 0$.

In a similar manner, we can establish Theorems 3.6.2 and 3.6.3 given below.

Theorem 3.6.2. *The solution of the fractional integral equation*

$$N(t) - N_0 t^{\mu-1} = -c^{\nu} \; {}_0D_t^{-\nu} \; N(t), \tag{3.6.15}$$

is given by

$$N(t) = N_0 \Gamma(\mu) t^{\mu-1} E_{\nu,\mu}(-c^{\nu} t^{\mu}), \tag{3.6.16}$$

where $E_{\nu,\mu}(\cdot)$ is the generalized Mittag-Leffler function (3.5.2), $\nu > 0, \mu > 0$.

Remark 3.6.3. When $\mu = 1$, we obtain the result given by Haubold and Mathai (2000).

Theorem 3.6.3. *If $c > 0, \nu > 0, \mu > 0$, then for the solution of the integral equation*

$$N(t) - N_0 \; t^{\mu-1} E_{\nu,\mu}^{\gamma}[-(ct)^{\nu}] = -c^{\nu} {}_0D_t^{-\nu} \; N(t), \tag{3.6.17}$$

there holds the formula

$$N(t) = N_0 \; t^{\mu-1} E_{\nu,\mu}^{\gamma+1}[-(ct)^{\nu}]. \tag{3.6.18}$$

Hint: Use the formula

$$L^{-1}\left\{s^{-\beta}(1-as^{-\alpha})^{-\gamma}\right\} = t^{\beta-1}E^{\gamma}_{\alpha,\beta}(at^{\alpha}), \tag{3.6.19}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(s) > |a|^{\frac{1}{\Re(\alpha)}}, \Re(s) > 0$.

Corollary 3.6.1. If $c > 0, \mu > 0, \nu > 0$, then for the solution of

$$N(t) - N_0 t^{\mu-1}E_{\nu,\mu}[-c^{\nu}t^{\nu}] = -c^{\nu}\ {}_0D_t^{-\nu}\ N(t), \tag{3.6.20}$$

there holds the relation

$$N(t) = \frac{N_0}{\nu}t^{\mu-1}\left[E_{\nu,\mu-1}(-c^{\nu}t^{\nu}) + (1+\nu-\mu)E_{\nu,\mu}(-c^{\nu}t^{\nu})\right]. \tag{3.6.21}$$

Theorem 3.6.4. *The Cauchy problem for the integro-differential equation*

$${}_0D_x^{\mu}\ f(x) + \lambda\ {}_0D_x^{-\nu}\ f(x) = h(x),\ \ (\lambda, \mu, \nu \in \mathbb{C}) \tag{3.6.22}$$

with the initial condition

$$D_x^{\mu-k-1}\ f(0) = a_k, k = 0, 1, \cdots, [\mu], \tag{3.6.23}$$

where $\Re(\nu) > 0, \Re(\mu) > 0$ and $h(x)$ is any integrable function on the finite interval [0,b] has the unique solution, given by

$$\begin{aligned} f(x) = &\int_0^x (x-t)^{\mu-1}E_{\mu+\nu,\mu}[-\lambda(x-t)^{\mu+\nu}]h(t)\mathrm{d}t \\ &+ \sum_{k=0}^{n-1} a_k x^{\alpha-k-1}E_{\mu+\nu,\mu-k}(-\lambda x^{\mu+\nu}) \end{aligned} \tag{3.6.24}$$

Proof 3.6.2. Exercise.

Theorem 3.6.5. *The solution of the equation*

$${}_0D_t^{\frac{1}{2}}\, f(t) + bf(t) = t > 0; \left[{}_0D_t^{-\frac{1}{2}}\, f(t)\right]_{t=0} = C, \tag{3.6.25}$$

where C is a constant is given by

$$f(t) = C\, t^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(-bt^{\frac{1}{2}}), \tag{3.6.26}$$

where $E_{\frac{1}{2},\frac{1}{2}}(\cdot)$ *is the Mittag-Leffler function.*

Proof 3.6.3. Exercise.

Remark 3.6.4. Theorem 3.6.5 gives the generalized form of the equation solved by Oldham and Spanier (1974).

Exercises 3.6.

3.6.1. Prove that if $c > 0, \nu > 0, \mu > 0$, then the solution of

$$N(t) - N_0 t^{\mu-1} E^2_{\nu,\mu}(c^\nu t^\nu) = -c^\nu {}_0D_t^{-\nu}\, N(t), \tag{3.6.27}$$

is given by

$$\begin{aligned} N(t) = N_0 t^{\mu-1} E^3_{\nu,\mu}(-c^\nu t^\nu) = \frac{N_0 t^{\mu-1}}{2\nu^2}\Big[& E_{\nu,\mu-2}(-c^\nu t^\nu) \\ & + \{3(\nu+1) - 2\mu\}\, E_{\nu,\mu-1}(-c^\nu t^\nu) \\ & + \left\{2\nu^2 + \mu^2 + 3\nu - 2\mu - 3\nu\mu + 1\right\} E_{\nu,\mu}(-c^\nu t^\nu)\Big], \end{aligned} \tag{3.6.28}$$

where $\Re(\nu) > 0, \quad \Re(\mu) > 2$.

3.6.2. Prove that if $\nu > 0, c > 0, d > 0, \mu > 0, c \neq d$, then for the solution of the equation

$$N(t) - N_0 t^{\mu-1} E_{\nu,\mu}(-d^\nu t^\nu) = -c^\nu {}_0D_t^{-\nu}\, N(t), \tag{3.6.29}$$

there holds the formula.

$$N(t) = N_0 \frac{t^{\mu-\nu-1}}{c^\nu - d^\nu}\left[E_{\nu,\mu-\nu}(-d^\nu t^\nu) - E_{\nu,\mu-\nu}(-c^\nu t^\nu)\right]. \tag{3.6.30}$$

3.6.3. Prove that if $c > 0, \nu > 0, \mu > 0$, then for the solution of the equation

$$N(t) - N_0 t^{\mu-1} E_{\nu,\mu}(-c^\nu t^\nu) = -c^\nu {}_0D_t^{-\nu}\, N(t), \tag{3.6.31}$$

the following result holds:

$$N(t) = \frac{N_0}{\nu} t^{\mu-1} \left[E_{\nu,\mu-1}(-c^\nu t^\nu) + (1 + \nu - \mu) E_{\nu,\mu}(-c^\nu t^\nu) \right]. \tag{3.6.32}$$

3.6.4. Solve the equation

$$ {}_0D_t^Q \ f(t) + {}_0D_t^q \ f(t) = g(t),$$

where $q - Q$ is not an integer or a half integer and the initial condition is

$$\left[{}_0D_t^{q-1} \ f(t) + {}_0D_t^{Q-1} \ f(t) \right]_{t=0} = C \tag{3.6.33}$$

where C is a constant.
3.6.5. Solve the equation

$$ {}_0D_t^\alpha \ x(t) - \lambda x(t) = h(t), \quad (t > 0), \tag{3.6.34}$$

subject to the initial conditions

$$\left[{}_0D_t^{\alpha-k} \ h(t) \right]_{t=0} = b_k, \quad (k = 1, \cdots, n) \tag{3.6.35}$$

where $n - 1 < \alpha < n$.

3.6.6 Prove Theorem 3.6.4

3.6.6 Prove Theorem 3.6.5.

3.6.2. Fractional diffusion

Theorem 3.6.6. *The solution of the following initial value problem for the fractional diffusion equation in one dimension*

$$ {}_0D_t^\alpha \ U(x,t) = \lambda^2 \frac{\partial^2 U(x,t)}{\partial x^2}, \quad (t > 0, -\infty < x < \infty) \tag{3.6.36}$$

with initial conditions :

$$\lim_{x\to\pm\infty} U(x,t) = 0; \left[{}_0D_t^{\alpha-1}\, U(x,t)\right]_{t=0} = \phi(x) \tag{3.6.37}$$

is given by

$$U(x,t) = \int_{-\infty}^{\infty} G(x-\zeta,t)\phi(\zeta)\mathrm{d}\zeta, \tag{3.6.38}$$

where

$$G(x,t) = \frac{1}{\pi}\int_0^{\infty} t^{\alpha-1}E_{\alpha,\alpha}(-k^2\lambda^2 t^{\alpha})\cos kx\,\mathrm{d}k. \tag{3.6.39}$$

Solution 3.6.1. Let $0 < \alpha < 1$. Using the boundary conditions (3.6.37), the Fourier transform of (3.6.36) with respect to variable x gives

$$ {}_0D_x^{\alpha}\,\bar{U}(k,t) + \lambda^2k^2\bar{U}(k,t) = 0 \tag{3.6.40}$$

$$\left[{}_0D_t^{\alpha-1}\,\bar{U}(k,t)\right]_{t=0} = \bar{\phi}(k), \tag{3.6.41}$$

where k is a Fourier transform parameter and ‘ $-$ ’ indicates Fourier transform. Applying the Laplace transform to (3.6.40) and using (3.6.41), it gives

$$\tilde{\bar{U}}(k,s) = \frac{\bar{\phi}(k)}{s^{\alpha}+k^2\lambda^2}, \tag{3.6.42}$$

where ‘ $\sim$ ’ indicates Laplace transform. The inverse Laplace transform of (3.6.42) yields

$$\bar{U}(k,t) = t^{\alpha-1}\bar{\phi}(k)E_{\alpha,\alpha}(-\lambda^2k^2t^2), \tag{3.6.43}$$

and then the solution is obtained by taking inverse Fourier transform. By taking inverse Fourier transform of (3.6.43) and using the formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-ikx}f(k)\mathrm{d}k = \frac{1}{\pi}\int_0^{\infty} f(k)\cos(kx)\mathrm{d}k \tag{3.6.44}$$

we have

$$U(x,t) = \int_{-\infty}^{\infty} G(x-\zeta,t)\phi(\zeta)\mathrm{d}\zeta, \tag{3.6.45}$$

where

$$G(x,t) = \frac{1}{\pi}\int_{0}^{\infty} t^{\alpha-1}E_{\alpha,\alpha}(-k^2\lambda^2 t^\alpha)\cos(kx)\mathrm{d}k \tag{3.6.46}$$

with $\Re(\alpha) > 0, k > 0$.

Exercise 3.6.

3.6.8 Evaluate the integral in (3.6.46).

3.6.9 Find the solution of the Fick's diffusion equation

$$\frac{\partial}{\partial t}P(x,t) = \lambda\frac{\partial^2}{\partial x^2}P(x,t),$$

with the initial condition $P(x,t=0) = \delta(x)$, where $\delta(x)$ is the Dirac delta function.

References

Caputo, M. (1969). *Elasticitá e Dissipazione*, Zanichelli, Bologna.

Dzherbashyan, M.M. (1966). *Integral Transforms and Representation of Functions in Complex Domain* (in Russian), Nauka, Moscow.

Erdélyi, A. (1950-51). On some functional transformations, *Univ. Politec. Torino, Rend. Sem. Mat.* **10**, 217-234.

Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G. (1953). *Higher Transcendental Functions*, Vol. 1, McGraw - Hill, New York, Toronto and London.

Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G. (1954). *Tables of Integral Transforms*, Vol. 1, McGraw - Hill, New York, Toronto and London.

Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G. (1954a). *Tables of Integral Transforms*, Vol. 2, McGraw - Hill, New York, Toronto and London.

Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G. (1955). *Higher Transcendental Functions*, Vol. 3, McGraw - Hill, New York, Toronto and London.

Fox, C. (1963). Integral transforms based upon fractional integration, *Proc. Cambridge Philos. Soc.*, **59**, 63-71.

Haubold, H. J. and Mathai, A. M. (2000). The fractional kinetic equation and thermonuc1ear functions, *Astrophysics and Space Science*, **273**, 53-63.

Hilfer, R., (Ed.). (2000). *Applications of Fractional Calculus in Physics*, World Scientific, Singapore.

Kalla, S. L. and Saxena, R. K. (1969). Integral operators involving hypergeometric functions, *Math. Zeitschr.*, **108**, 231-234.

Kilbas. A. A. and Saigo, M. (1998). Fractional calculus of the H-function, *Fukuoka Univ. Science Reports*, **28**, 41-51.

Kilbas, A. A. and Saigo, M. (1996). On Mittag- Leffler type function, fractional calculus operators and solutions of integral equations, *Integral Transforms and Special Functions*, **4**, 355-370.

Kilbas, A. A, Saigo, M. and Saxena, R. K. (2002). Solution of Volterra integrodifferential equations with generalized Mittag-Leffler function in the kernels, *J. Integral Equations and Applications*, **14**, 377-396

Kilbas, A. A., Saigo, M. and Saxena, R. K. (2004). Generalized Mittag-Leffler function and generalized fractional calculus operators, *Integral Transforms and Special Functions*, **15**, 31-49.

Kober, H. (1940). On fractional integrals and derivatives, *Quart. J. Math. Oxford*, Ser. ll, 193 -211.

Love, E. R (1967). Some integral equations involving hypergeometric functions, *Proc. Edin. Math. Soc.*, **15(2)**, 169-198.

Mathai, A. M. and Saxena, R. K. (1973). *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences*, Lecture Notes in Mathematics, **348**, Springer- Verlag, Berlin, Heidelberg.

Mathai, A. M. and Saxena, R. K. (1978). *The H-function with Applications in Statistics and Other Disciplines*, John Wiley and Sons, New York - London - Sydney.

Miller, K. S. and Ross, B. (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York.

Mittag-Leffler, G. M. (1903). Sur la nouvelle fonction $E_\alpha(x)$, *C. R. Acad. Sci. Paris*, (Ser. II) **137**, 554-558.

Oldham, K. B. and Spanier, J. (1974). *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic Press, New York.

Podlubny, 1. (1999). *Fractional Differential Equations*, Academic Press, San Diego.

Podlubny, 1. (2002). Geometric and physical interpretations of fractional integration and fractional differentiation, *Frac. Calc. Appl. Anal.*, **5(4)**, 367-386.

Prabhakar, T. R. (1971). A singular integral equation with a generalized Mittag-Leffler function in the kernel, *Yokohama Math. J.*, **19**, 7-15.

Ross, B. (1994). A formula for the fractional integration and differentiation of $(a + bx)^c$, *J. Fract. Calc.*, **5**, 87-89.

Saigo, M. (1978). A remark on integral operators involving the Gauss hypergeometric function, *Math. Reports of College of Gen. Edu. , Kyushu University*, **11**, 135 -143.

Saigo, M. and Raina, R. K. (1988). Fractional calculus operators associated with a general class of polynomials, *Fukuoka Univ. Science Reports*, **18**, 15 - 22.

Samko, S. G. , Kilbas, A. A. and Marichev, 0. 1. (1993). *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Reading.

Saxena, R. K. (1967). On fractional integration operators, *Math. Zeitsch.*, **96**, 288-291.

Saxena, R. K. (2002). Certain properties of generalized Mittag-Leffler function, *Proceedings of the Third Annual Conference of the Society for Special Functions and Their Applications*, Varanasi, March 4-6, 75-81.

Saxena, R. K. (2003). Alternative derivation of the solution of certain integro-differential equations of Volterra-type, *Ganita Sandesh,* **17(1)**, 51-56.

Saxena,R. K. (2004). On a unified fractional generalization of free electron laser equation, *Vijnana Parishad Anusandhan Patrika,* **47(l)**, 17-27.

Saxena, R. K. and Kumbhat, R. K. (1973). A generalization of Kober operators, *Vijnana Parishad Anusandhan Patrika*, **16**, 31-36.

Saxena, R. K. and Kumbhat, R. K. (1974). Integral operators involving H-function, *Indian J. Pure appl. Math.*, **5**, 1-6.

Saxena, R. K. and Kumbhat, R. K. (1975). Some properties of generalized Kober operators, *Vijnana Parishad Anusandhan Patrika*, **18**, 139-150.

Saxena, R. K, Mathai, A. M and Haubold, H. J. (2002). On fractional kinetic equations, *Astrophysics and Space Science*, **282**, 281-287.

Saxena, R. K, Mathai, A. M and Haubold, H. J. (2004). On generalized fractional kinetic equations, *Physica A* , **344**, 657-664.

Saxena, R. K, Mathai, A. M. and Haubold, H. J. (2004). Unified fractional kinetic equations and a fractional diffusion equation, *Astrophysics and Space Science*, **290**, 241-245.

Saxena, R. K. and Nishimoto, K. (2002). On a fractional integral formula of Saigo operator, *J. Fract. Calc.*, **22**, 57-58.

Saxena, R. K. and Saigo, M. (2005). Certain properties of fractional calculus operators associated with generalized Mittag-Leffler function. *Frac. Calc. Appl. Ana1.*, **8(2)**, 141-154.

Sneddon, I. N. (1975). *The Use in Mathematical Physics of Erdélyi- Kober Operators and Some of Their Applications*, Lecture Notes in Mathematics (Edited by B. Ross), **457**, 37-79.

Srivastava, H. M. and Saxena, R. K. (2001). Operators of fractional integration and their applications, *Appl. Math. Comput.*, **118**, 1- 52.

Stein, E. M. (1970). *Singular Integrals and Differential Properties of Functions*, Princeton University Press, New Jersey.

Wiman, A. (1905). Uber den Fundamental satz in der Theorie de Funktionen $E_\alpha(x)$. *Acta Math.*, **29**, 191-201

Weyl, H. (1917). Bemerkungen zum Begriff des Differentialquotienten gebrochener Ordnung, *Vierteljahresschr. Naturforsch. Gen. Zurich,* **62**, 296-302.