CHAPTER 4

APPLICATIONS IN ASTROPHYSICS

[*This chapter is based on the lectures of Professor Dr. Hans Haubold of the Outer Space Division of the United Nations.*]

4.0. Introduction

Since the participants of the Fourth S.E.R.C. school are of different backgrounds it is not worth discussing the individual topics in astrophysics. Only the mathematical aspects where the various special functions come in naturally will be discussed here. We start with a few topics in astrophysics such as solar models, energy generations in stars, gravitational instability problems and reaction-diffusion problems.

4.1. Solar Models

When looking at the internal structure of the Sun one has to look at mass conservation, hydrostatic equilibrium, energy conservation and energy transport. These can be described by a system of non-linear differential equations which cannot be fully solved analytically. Hence the standard techniques adopted are numerical evaluations thereby resulting in computer-generated solar models. In order to derive analytic models a starting point would be to look into the density distribution in the core of the Sun. It is well known that the matter density decreases from the center to the exterior, temperature and pressure also behave the same way. The solar core is a more stabilized region and hence an analytic model for the matter density distribution in the solar core is an appropriate starting point. Let r be an arbitrary distance from the center of the Sun and $R_\odot$ the solar radius. The simplest model for matter density $\rho(r)$ is a linear model of the type

$$\rho(r) = \rho_0 \left[1 - \frac{r}{R_\odot}\right] \qquad \text{(Model 1)}$$

which indicates that the matter density decreases linearly from the core to the surface and it is zero at the surface and it is a constant ρ_0 at the center. But observations indicate that a linear model is not correct. A more appropriate starting point would be to consider the non-linear model

$$\rho(r) = \rho_0 \left[1 - \left(\frac{r}{R_\odot}\right)^\delta\right], \delta > 0 \quad \text{(Model 2)} \tag{4.1.1}$$

where δ is an arbitrary parameter. Since the surface area of a sphere of radius r is $4\pi r^2$ the mass of the Sun can be computed from the relation.

$$\frac{\mathrm{d}}{\mathrm{d}r} M(r) = 4\pi r^2 \rho(r). \tag{4.1.2}$$

That is,

$$\begin{aligned} M(r) &= 4\pi\rho_0 \int_0^r t^2 \left[1 - \left(\frac{t}{R_\odot}\right)^\delta\right] \mathrm{d}t \\ &= \frac{4\pi}{3}\rho_0 r^3 \left[1 - \frac{3}{(\delta+3)}\left(\frac{r}{R_\odot}\right)^\delta\right]. \end{aligned} \tag{4.1.3}$$

Denoting the total mass by$M\odot$, this gives the central density

$$\rho_0 = \frac{3}{4\pi}\frac{\delta+3}{\delta}\frac{M_\odot}{R_\odot}. \tag{4.1.4}$$

From the connection between pressure $P(r)$, mass $M(r)$ and density $\rho(r)$, namely,

$$\frac{\mathrm{d}}{\mathrm{d}r} P(r) = G\frac{M(r)\rho(r)}{r^2} \tag{4.1.5}$$

where G is the gravitational constant, we have

$$
\begin{aligned}
P(r) &= P(0) - G\int_0^r \frac{M(t)\rho(t)}{t^2}\mathrm{d}t \\
&= \frac{4\pi G}{3}\rho_0^2 R_\odot^2\left\{\xi - \frac{1}{2}\left(\frac{r}{R_\odot}\right)^2 + \frac{(\delta+6)}{(\delta+2)(\delta+3)}\left(\frac{r}{R_\odot}\right)^{\delta+2}\right. \\
&\quad \left. - \frac{3}{2(\delta+1)(\delta+3)}\left(\frac{r}{R_\odot}\right)^{2\delta+2}\right\}
\end{aligned} \tag{4.1.6}
$$

where

$$
\xi = \frac{1}{2} - \frac{(\delta+6)}{(\delta+2)(\delta+3)} + \frac{3}{2(\delta+1)(\delta+3)}. \tag{4.1.7}
$$

From the relationship between temperature $T(r)$ and pressure $P(r)$, namely,

$$
T(r) = \frac{\mu}{kN_A}\frac{P(r)}{\rho(r)} \tag{4.1.8}
$$

where μ is the mean molecular weight, N_A is Avogadro's constant and k is the Boltzmann constant, we have

$$
\begin{aligned}
T(r) = \frac{4\pi G\mu\rho_0}{3kN_A}\frac{R_\odot^2}{[1-(\frac{r}{R_\odot})^\delta]}&\left\{\xi - \frac{1}{2}\left(\frac{r}{R_\odot}\right)^2\right. \\
&\left. + \frac{(\delta+6)}{(\delta+2)(\delta+3)}\left(\frac{r}{R_\odot}\right)^{\delta+2} - \frac{3}{2(\delta+1)(\delta+3)}\left(\frac{r}{R_\odot}\right)^{2\delta+2}\right\}.
\end{aligned} \tag{4.1.9}
$$

From the equation of energy conservation, namely,

$$
\frac{\mathrm{d}}{\mathrm{d}r}L(r) = 4\pi r^2\rho(r)\epsilon(r) \tag{4.1.10}
$$

where $L(r)$ represents the energy flux through the sphere with radius r so that $L(R_\odot)$ represents the luminosity of the Sun and $\epsilon(r)$ is the rate of thermonuclear energy generation per unit mass including the tiny energy losses via solar neutrinos, we can compute luminosity once an expression is available for $\epsilon(r)$.

If we consider a specific reaction, say particles 1 and 2 reacting to give rise to particles 3 and 4 or $1 + 2 \to 3 + 4$ then the internal luminosity due to this specific reaction is given by

$$L_{12}(R_\odot) = \int_0^{R_\odot} 4\pi r^2 \rho(r)\epsilon_{12}(r)\mathrm{d}r \tag{4.1.11}$$

A general model that can be used to represent $\epsilon(r)$ is the following:

$$\epsilon(r) = \epsilon_0 \left[\frac{\rho(r)}{\rho_0}\right]^\alpha \left[\frac{T(r)}{T_0}\right]^\beta, \tag{4.1.12}$$

where α and β are real constants.

4.1.1. A more general model for density

A more general two-parameter model for the matter density distribution is the following:

$$\rho(r) = \rho_0 \left[1 - \left(\frac{r}{R_\odot}\right)^\delta\right]^\gamma, \ \delta > 0, \gamma > 0 \quad \text{(Model 3)} \tag{4.1.13}$$

Example 4.1.1. Evaluate the total mass $M(R_\odot)$ as well as the mass contained in a sphere of arbitrary radius r under the model in (4.1.13).

Solution 4.1.1. The total mass is given by

$$\begin{aligned} M_\odot &= \int_0^{R_\odot} 4\pi r^2 \rho_0 \left[1 - \left(\frac{r}{R_\odot}\right)^\delta\right]^\gamma \mathrm{d}r &(4.1.14)\\ &= 4\pi R_\odot^3 \rho_0 \int_0^1 x^2 \left[1 - x^\delta\right]^\gamma \mathrm{d}x,\, x = \frac{r}{R_\odot} \\ &= 4\pi R_\odot^3 \rho_0 \int_0^1 y^{\frac{3}{\delta}-1} \left[1 - y\right]^\gamma \mathrm{d}y,\, y = x^\delta \\ &= 4\pi R_\odot^3 \rho_0 \frac{\Gamma\left(\frac{3}{\delta}\right)\Gamma(\gamma+1)}{\Gamma(\gamma + 1 + \frac{3}{\delta})}, &(4.1.15) \end{aligned}$$

evaluating (4.1.14) with the help of a type-1 beta integral. Mass at an arbitrary radius r is given by

$$M(r) = \int_0^r 4\pi t^2 \rho_0 \left[1 - \left(\frac{t}{R_\odot}\right)^\delta\right]^\gamma \mathrm{d}t$$
$$= 4\pi\rho_0 R_\odot^3 \int_0^{\frac{r}{R_\odot}} u^2[1 - u^\delta]^\gamma \mathrm{d}u, u = \frac{t}{R_\odot}, 0 \le u \le 1.$$

Expanding $(1 - \mu^\delta)^\gamma$ with the help of a binomial expansion and then integrating out we have the following:

$$(1 - \mu^\delta)^\gamma = (1 - \mu^\delta)^{-(-\gamma)} = \sum_{k=0}^\infty \frac{(-\gamma)_k}{k!}(u^\delta)^k.$$
$$M(r) = 4\pi\rho_0 R_\odot^3 \sum_{k=0}^\infty \frac{(-\gamma)_k}{k!} \int_0^{\frac{r}{R_\odot}} u^{2+\delta k} \mathrm{d}u$$
$$= 4\pi\rho_0 R_\odot^3 \sum_{k=0}^\infty \frac{(-\gamma)_k}{k!} \frac{(\frac{r}{R_\odot})^{3+\delta k}}{3 + \delta k}$$

But

$$\frac{1}{k + \frac{3}{\delta}} = \frac{1}{\frac{3}{\delta}} \frac{(\frac{3}{\delta})_k}{(\frac{3}{\delta} + 1)_k}.$$

Hence

$$M(r) = \frac{4\pi\rho_0}{3} \, {}_2F_1(-\gamma, \frac{3}{\delta}; \frac{3}{\delta} + 1; (\frac{r}{R_\odot})^\delta).$$

Note that the models in (4.1.13) and (4.1.1) are really valid only in the interior core of the Sun. The convective zone has entirely a different behavior for the density distribution. But for computing the total mass, pressure, temperature and luminosity we have integrated out over the entire length of the solar radius $R_\odot$. This is not appropriate. The integration should have been done only in the interior core of the Sun. Hence a more appropriate model is of the following form:

$$\rho(r) = \rho_0 \left[1 - a\left(\frac{r}{R_\odot}\right)^\delta\right]^\gamma, \quad \delta > 0, \gamma > 0, a > 0 \qquad (\text{ Model 4 }). \qquad (4.1.16)$$

Since $1 - ax^{\delta} > 0, x = \frac{r}{R_{\odot}}$, we have $0 \leq x \leq \frac{1}{a^{\frac{1}{\delta}}}$. Hence for the total integral the range should have been $0 \leq r \leq \frac{R_{\odot}}{a^{\frac{1}{\delta}}}$.

Exercises 4.1.

4.1.1. Evaluate ρ_0 in terms of the total mass $M_{\odot}$ for the Models 1, 2, 3 and 4.

4.1.2. Evaluate $M(r)$ for the Models 4 in (4.1.17).

4.1.3. Evaluate the expression for pressure $P(r)$ under Models 3 for $\delta = 2$.

4.1.4. Evaluate the expression for temperature under Models 3 for $\delta = 2$.

4.1.4. Evaluate the expression for $M(r)$ under Models 4.

4.2. Solar Thermonuclear Energy Generation

In reaction rate theory when two particles 1 and 2 reacting to give rise to particles 3 and 4, namely $1 + 2 \to 3 + 4$, the basic assumption is that the distribution of the relative velocities of the reacting particles always remains Maxwell-Boltzmannian. Then the distribution of the relative velocities of the particles can be written as

$$f(v)\mathrm{d}v = \left(\frac{\mu}{2\pi kT}\right)^{\frac{3}{2}} \exp\left\{-\frac{\mu v^2}{2kT}\right\} 4\pi v^2 \mathrm{d}v \tag{4.2.1}$$

such that $\int_0^{\infty} f(v)\mathrm{d}v = 1$. In terms of energy $E = \frac{\mu v^2}{2}$ we have the density for E given by,

$$f(E)\mathrm{d}E = \frac{2}{\pi^{\frac{1}{2}}(kT)^{\frac{3}{2}}} \exp\left\{-\frac{E}{kT}\right\} E^{\frac{1}{2}} \mathrm{d}E. \tag{4.2.2}$$

If the relative velocity of the interacting particles 1 and 2 is v then the thermally averaged product of the cross section σ, denoted by the expected value of σv or $< \sigma v >$ has the following expression under Maxwell-Boltzmann velocity distribution (see Mathai and Haubold (1998)) and in the charged particle case.

$$< \sigma v >= \left(\frac{8}{\pi\mu}\right)^{\frac{1}{2}} \sum_{\nu=0}^{2} \frac{S^{(\nu)}(0)}{\nu!} \frac{1}{(kT)^{-\nu+\frac{1}{2}}} \int_0^\infty y^\nu \mathrm{e}^{-y-zy^{-\frac{1}{2}}} \mathrm{d}y. \tag{4.2.3}$$

The reaction probability integral coming from $< \sigma v >$, denoted by $N_\nu(z)$, is then given by

$$N_\nu(z) = \int_0^\infty y^\nu \mathrm{e}^{-y-zy^{-\frac{1}{2}}} \mathrm{d}y. \tag{4.2.4}$$

A more general integral, where (4.2.4) is a particular case, is already evaluated in Example 1.6.3 of Chapter 1. From there we note that

$$N_\nu(z) = \frac{1}{\pi^{\frac{1}{2}}} G_{0,3}^{3,0}\left[\frac{z^2}{4}\Big|_{0,\frac{1}{2},1+\nu}\right], \tag{4.2.5}$$

where $G(\cdot)$ is a Meijer's G-function. Equation (4.2.3) is the situation under non resonant reactions. But with depleted Maxwell-Boltzmann distribution the reaction probability integral changes to the following:

$$N_\nu(z;\delta) = \int_0^\infty y^\nu \mathrm{e}^{-y-zy^{-\frac{1}{2}}} \mathrm{e}^{-y^\delta} \mathrm{d}y. \tag{4.2.6}$$

With modified Maxwell-Boltzmann distribution the reaction probability integral has the following form:

$$N_\nu(z,d) = \int_0^{\mathrm{d}} y^\nu \mathrm{e}^{-y-zy^{-\frac{1}{2}}} \mathrm{d}y, \ \mathrm{d} < \infty. \tag{4.2.7}$$

In this case it is a fractional integral. In the resonant situation it can be shown (see Mathai and Haubold (1988)) that the reaction probability integral has the following form:

$$N(a,b,q,g) = \int_0^\infty \frac{\mathrm{e}^{-ay-qy^{-\frac{1}{2}}}}{(b-y)^2+g^2} \mathrm{d}y. \tag{4.2.8}$$

In the corresponding depleted case the above integral will be modified to the following:

$$N(a,b,q,g,\delta) = \int_0^\infty \frac{\mathrm{e}^{-ay-qy^{-\frac{1}{2}}-cy^\delta}}{(b-y)^2+g^2} \mathrm{d}y. \tag{4.2.9}$$

In the corresponding modified resonant case the integral will have the fractional form

$$N_d(a, b, q, g, \delta) = \int_0^d \frac{e^{-ay-qy^{-\frac{1}{2}}-cy^{\delta}}}{(b-y)^2+g^2} dy. \tag{4.2.10}$$

All these various cases and the related situations are considered in a series of papers by Haubold and Mathai, some of the earlier ones are available from Mathai and Haubold (1988). Some of the recent works from 1988 to 2005 will be mentioned in the lectures to be given by Professor Dr. Hans Haubold at the Fourth SERC School.

References

Chandrasekhar, S. (1967). *An Introduction to the Study of Stellar Structure*, Dover Publications, New York.

Mathai, A.M. and Haubold, H.J. (1988). *Modem Problems in Nuclear and Neutrino Astrophysics*, Akademie-Verlag, Berlin.