CHAPTER 5

RAMANUJAN'S THEORIES OF THETA AND ELLIPTIC FUNCTIONS - III

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5.0. Introduction

The present lectures are aimed at covering some introductory aspects of the Jacobian and Weiestrassian elliptic functions and the cubic elliptic functions implied in Ramanujan's works. The lectures are a sequel to earlier lectures delivered by the author in June - July 2000 and March - April 2005 SERC Schools, vide Publications 31 and 32 of Centre for Mathematical Sciences, Trivandrum and Pala Campuses respectively. It is hoped that the lectures will lead the audience / readers to further reading and research.

5.1. Basic Identity of Ramanujan and Weierstrassian Theory of Elliptic Functions

Following is an identity of Ramanujan which plays an important role in his development of elliptic function theory [9, 11, 13].

Theorem 5.1.1. *If* $q = e^{i\tau}$ *and* $0 < \text{Im}\,\theta < \text{Im}\,\tau$*, then*

$$\left[\frac{1}{4}\cot\frac{\theta}{2} + \sum_{1}^{\infty}\frac{q^n \sin n\theta}{1-q^n}\right]^2 = \left(\frac{1}{4}\cot\frac{\theta}{2}\right)^2 + \left(\frac{1}{2}\sum_{1}^{\infty}\frac{nq^n}{1-q^n}\right)$$

$$+\sum_{1}^{\infty}\left\{\frac{q^n}{(1-q^n)^2}-\frac{nq^n}{2(1-q^n)}\right\}\cos n\theta. \tag{5.1.1}$$

or, with $\xi = e^{i\theta}$,

$$\left(\sum_{-\infty}^{\infty}{}'\frac{\xi^n}{1-q^n}\right)^2 = -2\sum_{1}^{\infty}\frac{nq^n}{1-q^n}+\sum_{1}^{\infty}\frac{(n+1)(\xi^n+q^n\xi^{-n})}{(1-q^n)}-\sum_{1}^{\infty}\frac{(\xi^n+q^n\xi^{-n})}{(1-q^n)^2}. \tag{5.1.2}$$

Proof 5.1.1. The following proof of Ramanujan is elementary. [12, p.138].

On using the elementary identity

$$\cot\frac{\theta}{2}\sin n\theta = (1+\cos n\theta)+2\sum_{m=1}^{n-1}\cos m\theta$$

we have

$$\begin{aligned}
&\left[\frac{1}{4}\cot\frac{\theta}{2}+\sum_{1}^{\infty}\frac{q^n\sin n\theta}{1-q^n}\right]^2\\
&\quad=\left(\frac{1}{4}\cot\frac{\theta}{2}\right)^2+\frac{1}{2}\sum_{1}^{\infty}\frac{q^n\sin n\theta\cot\frac{\theta}{2}}{1-q^n}+\sum_{1}^{\infty}\sum_{1}^{\infty}\frac{q^{m+n}\sin m\theta\sin n\theta}{(1-q^m)(1-q^n)}\\
&\quad=\left(\frac{1}{4}\cot\frac{\theta}{2}\right)^2+\sum_{0}^{\infty}cn\cos n\theta
\end{aligned}$$

where

$$C_0=\frac{1}{2}+\sum_{1}^{\infty}\frac{q^n}{1-q^n}+\frac{1}{2}\sum_{1}^{\infty}\left(\frac{q^n}{1-q^n}\right)^2=\frac{1}{2}\sum_{1}^{\infty}\frac{nq^n}{1-q^n}$$

as required in (5.1.1). Further for $n \geq 1$,

$$cn = \frac{1}{2}\frac{q^n}{1-q^n} + \sum_1^\infty \frac{q^{n+r}}{1-q^{n+r}} + \sum_1^\infty \left(\frac{q^r}{1-q^r}\right)\left(\frac{q^{n+r}}{1-q^{n+r}}\right)$$
$$-\frac{1}{2}\sum_1^{n-1} \frac{q^r}{1-q^r}\frac{q^{n-r}}{1-q^{n-r}}$$

which reduces to the required expression in (5.1.1) on some manipulations. We omit the details.

Exercises 5.1.

5.1.1. Show in the proof of Theorem 5.1.1 that C_n indeed equals

$$\left\{\frac{q^n}{(1-q^n)^2} - \frac{nq^n}{2(1-q^n)}\right\}, n \geq 1.$$

Remark 5.1.1 (9, p135.)**.** Identity (5.1.1) is indeed equivalent to the following identity in the Weierstrassian elliptic function theory:

$$\left\{\zeta(\theta) - \frac{\eta_1 \theta}{\pi}\right\}^2 - p(\theta) = -\frac{1}{6} + 4\sum_1^\infty \frac{q^m \cos m\theta}{(1-q^m)^2} \tag{5.1.3}$$

where

$$\zeta(\theta) := \frac{1}{2}\cot\frac{\theta}{2} + 2\sum_1^\infty \frac{q^n \sin n\theta}{1-q^n} + \theta\left\{\frac{1}{12} - 2\sum_1^\infty \frac{q^n}{(1-q^n)^2}\right\}$$

or, with $z = e^{i\theta}$

$$\zeta(\theta) := \frac{1}{2i}\left\{\frac{1+z}{1-z} + 2\sum_{1}^{\infty}\frac{zq^n}{1-zq^n} - 2\sum_{1}^{\infty}\frac{z^{-1}q^n}{1-z^{-1}q^n}\right\}$$

$$+\,\theta\left\{\frac{1}{12} - 2\sum_{1}^{\infty}\frac{q^n}{(1-q^n)^2}\right\} \tag{5.1.4}$$

$$:= -i\rho_1(z) + \left(\frac{\theta}{12}\right)P(q) \tag{5.1.5}$$

and

$$p(\theta) := -\zeta'(\theta) = \frac{1}{4}\operatorname{cosec}^2\frac{\theta}{2} - 2\sum_{1}^{\infty}\frac{nq^n\cos n\theta}{1-q^n} + 2\sum_{1}^{\infty}\frac{q^n}{(1-q^n)^2} - \frac{1}{12}$$

$$= -z\rho_1'(z) - \frac{P}{12} \tag{5.1.6}$$

or

$$p(\theta) := -\sum_{-\infty}^{\infty}\frac{zq^n}{(1-zq^n)^2} + 2\sum_{1}^{\infty}\frac{q^n}{(1-q^n)^2} - \frac{1}{12},\quad z = e^{i\theta} \tag{5.1.7}$$

and

$$\eta_1 := \pi\left\{\frac{1}{12} - 2\sum_{1}^{\infty}\frac{q^n}{(1-q^n)^2}\right\} = \zeta(\pi).$$

Definition 5.1.1. The functions $p(\theta)$ and $\zeta(\theta)$ are are respectively the Weierstrassian elliptic function and the Weierstrassian zeta function.

Exercises 5.1.

5.1.2. Prove double periodicity of $p(\theta)$ and identify the singularities.

5.1.3. Setting $\eta_2 = \zeta(\pi\tau)$, show that $\eta_2 = \tau\eta_1 - \frac{i}{2}$ (Legendre's formula).

Theorem 5.1.2. *We can rewrite $p(\theta)$ as*

$$p(\theta) = \frac{1}{\theta^2} + \sum_{m,n}{}' \left[\frac{1}{(\theta - 2\pi n - 2\pi\tau m)^2} - \frac{1}{(2\pi n + 2\pi\tau m)^2}\right] \tag{5.1.8}$$

with $q = \mathrm{e}^{2\pi i\tau}$, which is the classical form of Weierstrassian elliptic function.

Proof 5.1.2. We have from Definition 5.1.1 that

$$\begin{aligned}
p(\theta) &= -\frac{1}{12} + 2\sum_{1}^{\infty} \frac{1}{(\mathrm{e}^{\pi i m\tau} - \mathrm{e}^{-\pi i m\tau})^2} - \sum_{-\infty}^{\infty} \frac{1}{(\mathrm{e}^{i\frac{\theta}{2} + i\pi\tau n} - \mathrm{e}^{-\frac{i\theta}{2} - i\pi\tau n})^2} \\
&= -\frac{1}{12} - \frac{1}{2}\sum_{1}^{\infty} \frac{1}{\sin^2 \pi m\tau} + \frac{1}{4}\sum_{-\infty}^{\infty} \frac{1}{\sin^2(\frac{\theta}{2} + \pi\tau m)} \\
&= \frac{1}{4\sin^2\frac{\theta}{2}} - \frac{1}{12} - \frac{1}{4}\sum_{-\infty}^{\infty}{}' \frac{1}{\sin^2 m\pi\tau} + \frac{1}{4}\sum_{-\infty}^{\infty} \frac{1}{\sin^2(\frac{\theta}{2} + m\pi\tau)} \\
&= \frac{1}{4\sin^2\frac{\theta}{2}} - \frac{1}{12} + \frac{1}{4}\sum_{-\infty}^{\infty}{}' \left(\frac{1}{\sin^2(\frac{\theta}{2} + m\pi\tau)} - \frac{1}{\sin^2 m\pi\tau}\right) \\
&= \frac{1}{4}\sum_{-\infty}^{\infty} \frac{1}{(\frac{\theta}{2} - \pi n)^2} - \frac{1}{12}\frac{3}{\pi^2}\sum_{-\infty}^{\infty}{}' \frac{1}{n^2} \\
&\quad + \frac{1}{4}\sum_{m=-\infty}^{\infty}{}' \sum_{n=-\infty}^{\infty} \left[\frac{1}{(\frac{\theta}{2} + m\pi\tau - n\pi)^2} - \frac{1}{(m\pi\tau - n\pi)^2}\right] \\
&= \frac{1}{\theta^2} + \sum_{-\infty}^{\infty}{}' \left[\frac{1}{(\theta - 2\pi n)^2} - \frac{1}{(2\pi n)^2}\right] \\
&\quad + \sum_{m=-\infty}^{\infty}{}' \sum_{n=-\infty}^{\infty} \left[\frac{1}{(\theta - 2m\pi\tau - 2n\pi)^2} - \frac{1}{(2m\pi\tau + 2n\pi)^2}\right] \\
&= \frac{1}{\theta^2} + \sum_{-\infty}^{\infty} \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(\theta - 2m\pi\tau - 2n\pi)^2} - \frac{1}{(2m\pi\tau + 2n\pi)^2}\right).
\end{aligned}$$

This proves the theorem.

Theorem 5.1.3. [13] *(Generalization of basic identity of Ramanujan)*
Let $|q| < |z| < 1$ and let

$$\rho_1(z) := \frac{1}{2} + \sum_{-\infty}^{\infty}{}' \frac{z^n}{1-q^n}, \tag{5.1.9}$$

as before, and let

$$\rho_2(z) := -\frac{1}{12} + \sum_{-\infty}^{\infty}{}' \frac{q^n z^n}{(1-q^n)^2}. \tag{5.1.10}$$

Then

$$\rho_1(\alpha)\rho_1(\beta) + \rho_1(\beta)\rho_1(\gamma) + \rho_1(\gamma)\rho_1(\alpha) = \rho_2(\alpha) + \rho_2(\beta) + \rho_2(\gamma) \tag{5.1.11}$$

for all complex $\alpha,\ \beta$ and γ with $\alpha\beta\gamma = 1$.

Proof 5.1.3. We can rewrite $\rho_1(z)$ and $\rho_2(z)$ in their global forms on slight manipulations:

$$\rho_1(z) = \frac{1}{2}\left(\frac{1+z}{1-z}\right) + \sum_{1}^{\infty} \frac{q^n(z^n - z^{-n})}{1-q^n} \tag{5.1.12}$$

$$= \frac{1}{2}\left(\frac{1+z}{1-z}\right) + \sum_{1}^{\infty} \frac{q^n z}{1-q^n z} - \sum_{1}^{\infty} \frac{q^n z^{-1}}{1-q^n z^{-1}} \tag{5.1.13}$$

and

$$\rho_2(z) = -\frac{1}{12} + \sum_{1}^{\infty} \frac{q^n(z^n + z^{-n})}{(1-q^n)^2} \tag{5.1.14}$$

$$= -\frac{1}{12} + \sum_{1}^{\infty} \frac{nq^n z}{1-q^n z} + \sum_{1}^{\infty} \frac{nq^n z^{-1}}{1-q^n z^{-1}}. \tag{5.1.15}$$

With these global forms (5.1.11) would be valid globally except $\alpha\beta\gamma = 1$. We easily see from (5.1.12) and (5.1.14) that

$$\rho_1\left(\frac{1}{z}\right) = -\rho_1(z) \quad\text{and}\quad \rho_2\left(\frac{1}{z}\right) = \rho_2(z). \tag{5.1.16}$$

Employing (5.1.16) we can expand each side of (5.1.11) into power series in α and β and then see that the two sides are equal. We omit details.

Corollary 5.1.1. Identity (5.1.11) can be rewritten as

$$\rho_2(\alpha) + \rho_2(\beta) + \rho_2(\alpha\beta) = (1-\beta)\,\rho_1(\beta)\left\{\frac{\rho_1(\alpha\beta) - \rho_1(\alpha)}{\beta - 1}\right\} - \rho_1(\alpha\beta)\,\rho_1(\alpha).$$

Letting $\beta \to 1$ we get

$$2\rho_2(\alpha) + \rho_2(1) = \alpha\rho_1'(\alpha) - \rho_1^2(\alpha). \tag{5.1.17}$$

This is indeed the same as Ramanujan's basic identity (5.1.1).

Exercises 5.1.

5.1.4. Complete the details in the proof of Theorem 5.1.3.

5.1.5. Prove the equivalence of equations (5.1.17) and (5.1.1).

Theorem 5.1.4. *(Ramanujan's basic identity and addition theorem for the Weierstrassian elliptic function). The following holds*

$$p(a+b) = -p(a) - p(b) + \frac{1}{4}\left[\frac{p'(a) - p'(b)}{p(a) - p(b)}\right]^2. \tag{5.1.18}$$

Proof 5.1.4. From (5.1.17) (or what is the same (5.1.1)) and on using (5.1.11) we have, for $\alpha\beta\gamma = 1$,

$$(\rho_1(\alpha) + \rho_1(\beta) + \rho_1(\gamma))^2 = \sum \alpha\rho_1'(\alpha) - 3\rho_2(1) \tag{5.1.19'}$$

and hence, on differentiating with respect to β,

$$2\left(\sum \rho_1(\alpha)\right)\left(\rho_1'(\beta) - \frac{\gamma}{\beta}\rho_1'(\gamma)\right) = \beta\rho_1''(\beta) + \left[\gamma\rho_1''(\gamma) + \rho_1'(\gamma)\right]\left(-\frac{\gamma}{\beta}\right) + \rho_1'(\beta)$$

or,

$$2\left(\sum \rho_1(\alpha)\right)(\beta\rho_1'(\beta) - \gamma\rho_1'(\gamma)) = \beta^2\rho_1''(\beta) - \gamma^2\rho_1''(\gamma) - \gamma\rho_1'(\gamma) + \beta\rho_1'(\beta) \tag{5.1.19}$$

or,

$$\sum \rho_1(\alpha) = \frac{1}{2}\left[\frac{\beta^2\rho_1''(\beta) - \gamma^2\rho_1''(\gamma)}{\beta\rho_1'(\beta) - \gamma\rho_1'(\gamma)} + 1\right]. \tag{5.1.20}$$

Now, from (5.1.6),

$$\begin{aligned} p'(a) &= -i[\alpha^2\rho_1''(\alpha) + \alpha\rho_1'(\alpha)], \quad \alpha = e^{ia} \\ p'(b) &= -i[\beta^2\rho_1''(\beta) + \beta\rho_1'(\beta)], \quad \beta = e^{ib} \\ p'(c) &= -i[\gamma^2\rho_1''(\gamma) + \gamma\rho_1'(\gamma)], \quad \gamma = e^{ic}, \quad \alpha\beta\gamma = 1, \end{aligned}$$

so that,

$$p'(b) - p'(c) = -i[\beta^2\rho_1''(\beta) - \gamma^2\rho_1''(\gamma) + \beta\rho_1'(\beta) - \gamma\rho_1'(\gamma)].$$

Using (5.1.6) again, the last equation gives

$$\frac{p'(b) - p'(c)}{p(b) - p(c)} = \left[\frac{\beta^2\rho_1''(\beta) - \gamma^2\rho_1''(\gamma)}{\beta\rho_1'(\beta) - \gamma\rho_1'(\gamma)} + 1\right].$$

From (5.1.19') and (5.1.20), we have

$$
\begin{aligned}
\frac{1}{4}\left(\frac{p'(b)-p'(c)}{p(b)-p(c)}\right)^2 &= \left(\sum \rho_1(\alpha)\right)^2 \\
&= 3\rho_2(1) - \sum \alpha\rho_1'(\alpha) \\
&= 3\rho_2(1) - \alpha\rho_1'(\alpha) - \beta\rho_1'(\beta) - \gamma\rho_1'(\gamma) \\
&= p(a) + p(b) + p(c), \text{on using (5.1.6) and (5.1.10),} \\
&= p(-b-c) + p(b) + p(c) \\
&= p(b+c) + p(b) + p(c), \text{since} \quad p(\theta) \quad \text{is even by (5.1.8).}
\end{aligned}
$$

This is the same as (5.1.18) but for the arguments.

5.2. Basic Identity of Ramanujan and Jacobi's Elliptic Functions

Definition 5.2.1. Ramanujan [2][11, Second Notebook, Chapter 18] defines

$$
\begin{aligned}
\mathcal{S} &:= \sum_0^\infty \frac{\sin(2n+1)\frac{\theta}{2}}{\sinh(2n+1)\frac{y}{2}} \\
C &:= \sum_0^\infty \frac{\cos(2n+1)\frac{\theta}{2}}{\cosh(2n+1)\frac{y}{2}}
\end{aligned}
$$

and

$$
C_1 := \frac{1}{2} + \sum_1^\infty \frac{\cos n\theta}{\cosh ny}
$$

and are infact the Jacobian elliptic functions (in their Fourier series form) sn, cn and dn:

$$sn\left(\frac{1}{2}z\theta\right) := \frac{2}{z\sqrt{x}}\ S := \frac{2}{z\sqrt{x}}\sum_0^\infty \frac{\sin(2n+1)\frac{\theta}{2}}{\sinh(2n+1)\frac{y}{2}}$$

$$cn\left(\frac{1}{2}z\theta\right) := \frac{2}{z\sqrt{x}}\ C := \frac{2}{z\sqrt{x}}\sum_0^\infty \frac{\cos(2n+1)\frac{\theta}{2}}{\cosh(2n+1)\frac{y}{2}}$$

and

$$dn\left(\frac{1}{2}z\theta\right) := \frac{2}{z}\ C_1 := \frac{2}{z}\left[\frac{1}{2} + \sum_1^\infty \frac{\cos n\theta}{\cosh ny}\right]$$

or, in standard notations and complex form,

$$\begin{aligned}
sn\left(\frac{K}{\pi}\theta, k\right) &:= -\frac{i\pi}{Kk}e^{\frac{\pi i\tau}{2}+\frac{i\theta}{2}}\sum_{-\infty}^\infty \frac{(q\zeta)^n}{1-q^{2n+1}}\\
&= -\frac{i\pi}{Kk}e^{\frac{\pi i\tau}{2}+\frac{i\theta}{2}} f(q, q\zeta)\\
&= -\frac{i\pi}{Kk}e^{\frac{\pi i\tau}{2}+\frac{i\theta}{2}} f(e^{\pi i\tau}, e^{\pi i\tau+i\theta})\\
cn\left(\frac{K\theta}{\pi}, k\right) &:= \frac{\pi}{Kk}e^{\frac{\pi i\tau}{2}+\frac{i\theta}{2}} f(-q, q\zeta)\\
&= \frac{\pi}{Kk}e^{\frac{\pi i\tau}{2}+\frac{i\theta}{2}}\sum_{-\infty}^\infty \frac{(q\zeta)^n}{1+q^{2n+1}}\\
&= \frac{\pi}{Kk}e^{\frac{\pi i\tau}{2}+\frac{i\theta}{2}} f(-e^{\pi i\tau}, q^{\pi i\tau+i\theta})
\end{aligned}$$

and

$$dn\left(\frac{K\theta}{\pi}, k\right) = \frac{2}{z}\ C_1 = \frac{\pi}{K} f(-1, q\zeta) = \frac{\pi}{K}\sum_{-\infty}^\infty \frac{(q\zeta)^\eta}{1+q^{2n}} = \frac{\pi}{K} f(e^{i\pi}, e^{i\pi\tau+i\theta})$$

where

$$f(a,t) = \sum_{-\infty}^{\infty} \frac{t^n}{1-aq^{2n}}, \quad |q|<1, \quad a \neq q^{2n}, n = 0, \pm 1, \cdots$$

$$q = e^{i\pi\tau} = e^{-y},$$

$$\zeta = e^{i\theta}$$

$$x = k^2 := 16q(-q^2;q^2)_\infty^8/(-q;q^2)_\infty^8$$

$$\frac{2K}{\pi} := z := \left(\sum_{-\infty}^{\infty} q^{n^2}\right) = (-q;q^2)_\infty^4 (q^2;q^2)_\infty^2.$$

Here, as usual,

$$(a;q)_\infty := \Pi_0^\infty (1-aq^n), \quad |q|<1$$

$$(a;q)_n := (a;q)_\infty/(aq^n;q)_\infty$$

$$= \begin{cases} 1 & \text{if } n=0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{if } n \text{ is a positive integer.} \end{cases}$$

Theorem 5.2.1. *(Ramanujan) [11, Second Notebook, Chapter 18].*

$$C^2 + S^2 = \frac{xz^2}{4} \quad or \quad cn^2 + sn^2 = 1$$

$$C_1^2 + S^2 = \frac{z^2}{4} \quad or \quad dn^2 + k^2 sn^2 = 1$$

and

$$sncn = \frac{\pi^2}{K^2k^2} CS = \frac{\pi^2}{K^2k^2} \sum_1^\infty \frac{n \sin n\theta}{\cosh ny}.$$

$$CS + \frac{dC_1}{d\theta} = 0 = C_1 S + \frac{dC}{d\theta} = CC_1 - \frac{dS}{d\theta}.$$

Further, if for $0 \leq \phi < 2\pi$

$$\left(\frac{z\sqrt{x}}{2} cn = \right) C = \frac{z\sqrt{x}}{2} \cos\phi \ (i.e., \ cn\left(\frac{K}{\pi}\theta, k\right) = \cos\phi)$$

and

$$\left(\frac{z\sqrt{x}}{2}sn =\right) \quad S = \frac{z\sqrt{x}}{2}\sin\phi \quad \left(i.e., \ sn\left(\frac{K}{\pi}\theta, k\right) = \sin\phi\right).$$

then

$$\left(\frac{z}{2}dn =\right) \ C_1 = \frac{z}{2}\sqrt{1 - x\sin^2\phi}$$

that is

$$dn = \sqrt{1 - k^2 sn^2}$$

$$\frac{z}{2}\cos\phi\sqrt{1 - x\sin^2\phi} = \frac{\mathrm{d}}{\mathrm{d}\theta}\sin\phi = \cos\phi\frac{\mathrm{d}\phi}{\mathrm{d}\theta}$$

that is

$$\frac{K}{\pi}cn\sqrt{1 - k^2 sn^2} = \frac{\mathrm{d}}{\mathrm{d}\theta}sn\left(\frac{K}{\pi}\theta, k\right) = cn\left(\frac{K}{\pi}\theta, k\right)\frac{\mathrm{d}\phi}{\mathrm{d}\theta}$$

and

$$\theta = \frac{2}{z}\int_0^{\phi}\frac{\mathrm{d}\phi}{\sqrt{1 - x\sin^2\phi}}$$

that is

$$\frac{K}{\pi}\frac{\mathrm{d}\theta}{\mathrm{d}\phi} = \frac{1}{\sqrt{1 - k^2 sn^2}} = \frac{1}{dn} \quad or \quad \frac{\mathrm{d}\phi}{\mathrm{d}\theta} = \frac{K}{\pi}dn\left(\frac{K\theta}{\pi}, k\right).$$

Proof 5.2.1. We omit the proof but direct the readers to [13] and to the author's contributory Chapter to R.P. Agarwal's book [1]. We may just mention here that Ramanujan's basic identity (5.1.1) directly yields the first two identities while the others follow easily.

Exercises 5.2.

5.2.1. Learn the proof of Theorem 5.2.1 from [1. Chapter 5 by S. Bhargava].

Theorem 5.2.2. *The following product forms hold for the Jacobian elliptic functions C, S and C_1.*

$$S = \frac{Kk}{\pi} sn\left(\frac{K\theta}{\pi}, k\right) = \frac{-i(q\zeta)^{\frac{1}{2}}(q^2\zeta; q^2)_\infty(\zeta^{-1}; q^2)_\infty(q^2; q^2)_\infty^2}{(q\zeta; q^2)_\infty(q\zeta^{-1}; q^2)_\infty(q; q^2)_\infty^2}$$

$$C = \frac{Kk}{\pi} cn\left(\frac{K\theta}{\pi}, k\right) = \frac{(q\zeta)^{\frac{1}{2}}(-q^2\zeta; q^2)_\infty(-\zeta^{-1}; q^2)_\infty(q^2; q^2)_\infty^2}{(q\zeta; q^2)_\infty(q\zeta^{-1}; q^2)_\infty(-q; q^2)_\infty^2}$$

and

$$C_1 = \frac{K}{\pi} dn\left(\frac{K\theta}{\pi}, k\right) = \frac{(-q\zeta; q^2)_\infty(-q\zeta^{-1}; q^2)_\infty(q^2; q^2)_\infty^2}{2(q\zeta; q^2)_\infty(q\zeta^{-1}; q^2)_\infty(-q^2; q^2)_\infty^2}.$$

Proof 5.2.2. We recall the "remarkable" ${}_1\psi_1-$ summation formula of Ramanujan [9,11,13]

$$\sum_{-\infty}^{\infty} \frac{(\alpha^{-1}; q^2)_n(-\alpha qz)^n}{(\beta q^2; q^2)_n} = \sum_{-\infty}^{\infty} \frac{(\beta^{-1}; q^2)_n(-\frac{\beta q}{z})^n}{(\alpha q^2; q^2)_n}$$
$$= \frac{(-qz; q^2)_\infty(-\frac{q}{z}; q^2)_\infty(q^2; q^2)_\infty(\alpha\beta q^2; q^2)_\infty}{(-\alpha qz; q^2)_\infty(-\frac{\beta q}{z}; q^2)_\infty(\alpha q^2; q^2)_\infty(\beta q^2; q^2)_\infty}.$$

Putting $\alpha^{-1} = \beta$ and changing z to ζ in this we get

$$\sum_{-\infty}^{\infty} \frac{(-\frac{q\zeta}{\beta})^n}{1 - \beta q^{2n}} = \frac{(-q\zeta; q^2)_\infty(-q\zeta^{-1}; q^2)_\infty(q^2; q^2)_\infty^2}{(-\frac{q\zeta}{\beta}; q^2)_\infty(-\frac{\beta q}{\zeta}; q^2)_\infty(\beta q^2; q^2)_\infty(\frac{q^2}{\beta}; q^2)_\infty}.$$

Putting $\beta = q$, changing ζ to $-\zeta q$ in this and multiplying throughout by $-i(\zeta q)^{\frac{1}{2}}$ we get the first of the required identities. Similarly $\beta \to -q,\ \zeta \to \zeta q$ gives the second and $\beta \to -1$ gives the third.

Exercises 5.2.

5.2.2. Work out the details in the proof of Theorem 5.2.2.

5.2.3. With $f(a,t)$ as before, put (following S. Cooper [7])

$$f_1(\theta) := -i f(e^{i\pi}, e^{i\theta})$$
$$f_2(\theta) := -ie^{i\frac{\theta}{2}} f(e^{i\pi\tau}, e^{i\theta})$$

and

$$f_3(\theta) := -ie^{i\frac{\theta}{2}} f(e^{i\pi+i\pi\tau}, e^{i\theta}).$$

Then show that they are respectively the Jacobi's elliptic functions $\frac{K}{\pi}cs\left(\frac{K\theta}{\pi},k\right)$, $\frac{K}{\pi}ns\left(\frac{K\theta}{\pi},k\right)$ and $\frac{K}{\pi}ds\left(\frac{K\theta}{\pi},k\right)$. In other words, show that

$$\begin{aligned}
f_1(\theta) &= \frac{1}{2}\cot\frac{\theta}{2}\,\frac{(q^2;q^2)^2_\infty}{(-q^2;q^2)^2_\infty}\prod_{n=1}^{\infty}\frac{(1+2q^{2n}\cos\theta+q^{4n})}{(1-2q^{2n}\cos\theta+q^{4n})}\\
&= \frac{1}{2}\cot\frac{\theta}{2} - 2\sum_{1}^{\infty}\frac{q^{2n}}{1+q^{2n}}\sin n\theta\\
f_2(\theta) &= \frac{1}{2}\mathrm{cosec}\frac{\theta}{2}\frac{(q^2;q^2)^2_\infty}{(-q^2;q^2)^2_\infty}\prod_{n=1}^{\infty}\frac{(1-2q^{2n-1}\cos\theta+q^{4n-2})}{(1-2q^{2n}\cos\theta+q^{4n})}\\
&= \frac{1}{2}\mathrm{cosec}\frac{\theta}{2} + 2\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1-q^{2n+1}}\sin(n+\frac{1}{2})\theta\\
f_3(\theta) &= \frac{1}{2}\mathrm{cosec}\frac{\theta}{2}\frac{(q^2;q^2)^2_\infty}{(-q^2;q^2)^2_\infty}\prod_{n=1}^{\infty}\frac{(1+2q^{2n-1}\cos\theta+q^{4n-2})}{(1-2q^{2n}\cos\theta+q^{4n})}\\
&= \frac{1}{2}\mathrm{cosec}\frac{\theta}{2} - 2\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1+q^{2n+1}}\sin(n+\frac{1}{2})\theta.
\end{aligned}$$

5.2.4. Write out the product forms for S, C and C_1 obtained in Theorem 5.2.2 in respective trigonometric forms as in Exercise 5.2.3

5.3. Venkatachaliengar's Generalization of Ramanujan's Fundamental Identity and Relations Between Jacobian's and Weierstrassian Elliptic Functions

The following theorem is a generalization due to K. Venkatachaliengar [13] of Ramanujan's basic identity (5.1.1) and is crucial to further development of Ramunajan's theory of elliptic functions.

Theorem 5.3.1. *(Fundamental multiplicative identity)*
If $f(a,t)$ is as in Definition 5.2.1, we have

$$f(x,y)f(x,z) = x\frac{\partial}{\partial x}f(x,yz) + f(x,yz)(\rho_1(y)+\rho_2(z)) \tag{5.3.1}$$

ρ_1 *being as in (5.1.9).*

Proof 5.3.1. We have

$$\begin{aligned} f(x,y)f(x,z) &= \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\frac{y^n z^m}{(1-xq^{2n})(1-xq^{2m})} \\ &= \sum_{-\infty}^{\infty}\frac{(yz)^n}{(1-xq^{2n})^2} + \sum_{-\infty}^{\infty}\sum_{\substack{-\infty \\ m\neq n}}^{\infty}\frac{y^n z^m}{(1-xq^{2m})(1-xq^{2n})}. \end{aligned}$$

It is enough to show that the first sum and the second sum in the last identity equal respectively the first and second terms of the right side of the identity to be proved except perhaps for mutually canceling terms. This we do now. Firstly,

$$\begin{aligned} \sum_{-\infty}^{\infty}\frac{(yz)^n}{(1-xq^{2m})^2} &= \sum_{-\infty}^{\infty}\frac{\partial}{\partial x}\frac{\left(\frac{yz}{q^2}\right)^m}{(1-xq^{2m})} = \frac{\partial}{\partial x}\sum_{-\infty}^{\infty}\frac{\left(\frac{yz}{q^2}\right)^m}{(1-xq^{2m})} \\ &= \frac{\partial}{\partial x}f\left(\frac{yz}{q^2},x\right) = \frac{\partial}{\partial x}[xf(y,zx)] = x\frac{\partial}{\partial x}f(yz,x) + f(yz,x) \end{aligned}$$

on employing the easily proved identity

$$f(yzq^{-2},x) = xf(yz,x).$$

Finally,

$$\begin{aligned}
\sum_{m\neq n}\sum \frac{y^n z^m}{(1-xq^{2n})(1-xq^{2m})} &= \sum_{m=-\infty}^{\infty}\sum_k{}' \frac{y^{m+k}\ z^m}{(1-xq^{2m+2k})(1-xq^{2m})} \\
&= \sum_{-\infty}^{\infty}\sum_k{}' \frac{y^{m+k}\ z^m}{(1-q^{2k})(1-xq^{2m+2k})} + \sum_{-\infty}^{\infty}\sum_k{}' \frac{y^{m+k}\ z^m}{(1-q^{-2k})(1-xq^{2m})} \\
&= \sum_{-\infty}^{\infty}\sum_k{}' \frac{(yz)^{m+k}}{(1-xq^{2(m+k)})}\frac{z^{-k}}{(1-q^{2k})} + \sum_{-\infty}^{\infty}\sum_k{}' \frac{(yz)^m}{(1-xq^{2m})}\frac{y^k}{(1-q^{-2k})} \\
&= \left(-\frac{1}{2}+\rho_1(z^{-1})\right) f(yz,x) + \left(-\frac{1}{2}+\rho_1(y^{-1})\right) f(yz,x) \\
&= (\rho_1(y)+\rho_1(z))\ f(yz,x) - f(yz,x)
\end{aligned}$$

on changing $m+k$ to m in the first sum and k to $-k$ in the second sum and using the trivial property

$$\rho_1(z) = \rho_1(z^{-1}) \text{ of } \rho_1.$$

Corollary 5.3.1. [7]

$$f(e^{i\alpha},\ e^{i\theta})\ f(e^{-i\alpha},\ e^{i\theta}) = p(\alpha) - p(\theta). \tag{5.3.2}$$

Proof 5.3.2. Letting $y \to \frac{1}{z}$ in (5.3.1), we have

$$f(x,y)\ f(x,y^{-1}) = x\frac{\mathrm{d}}{\mathrm{d}x}\rho_1(x) - y\frac{\mathrm{d}}{\mathrm{d}y}\rho_1(y). \tag{5.3.3}$$

For,

$$\lim_{y\to\frac{1}{z}} x\frac{\partial}{\partial x} \sum_{-\infty}^{\infty} \frac{x^n}{1-yzq^{2n}} = \sum_{-\infty}^{\infty}{}' \frac{nx^n}{1-q^{2n}} = x\frac{\mathrm{d}}{\mathrm{d}x}\rho_1(x)$$

and

$$\lim_{y\to\frac{1}{z}} f(x,yz)\,(\rho_1(y)+\rho_1(z)) = \lim_{y\to\frac{1}{z}} (1-yz)\ f(yz,x)\ \lim_{y\to\frac{1}{z}}\left[\frac{\rho_1(y)+\rho_1(z)}{1-yz}\right]$$
$$= (-1)\lim_{y\to\frac{1}{z}}\left[\frac{\rho_1(y)-\rho_1(\frac{1}{z})}{y-\frac{1}{z}}\right]\frac{1}{z}$$
$$= -y\ \rho_1'(y).$$

Now (5.3.3) implies (5.3.2) on putting $x = e^{i\theta}$ and $y = e^{i\alpha}$ on using (5.1.6).

Corollary 5.3.2. [7]

$$\left(f_1^2(\theta) =\right)\ \frac{K^2}{\pi^2}cs^2\left(\frac{K\theta}{\pi},k\right) = p(\theta) - e_1 \tag{5.3.4}$$

$$\left(f_2^2(\theta) =\right)\ \frac{K^2}{\pi^2}ns^2\left(\frac{K\theta}{\pi},k\right) = p(\theta) - e_2 \tag{5.3.5}$$

$$\left(f_3^2(\theta) =\right)\ \frac{K^2}{\pi^2}ds^2\left(\frac{K\theta}{\pi},k\right) = p(\theta) - e_3 \tag{5.3.6}$$

where

$$e_1 = p(\pi),\ \ e_2 = p(\pi\tau)\ \text{ and }\ e_3 = p(\pi+\pi\tau).$$

Further,

$$e_1 - e_2 = \frac{1}{4}\ \frac{(-q;q^2)_\infty^4\ (q^2;q^2)_\infty^4}{(q;q^2)_\infty^4\ (-q^2;q^2)_\infty^4} = f_2^2(\theta) - f_1^2(\theta) = \frac{K^2}{\pi^2}\ (ns^2 - cs^2) \tag{5.3.7}$$

$$e_3 - e_2 = 4q\ \frac{(-q^2;q^2)_\infty^4 (q^2;q^2)_\infty^4}{(-q;q^2)_\infty^4\ (q;q^2)_\infty^4} = f_2^2(\theta) - f_3^2(\theta) = \frac{K^2}{\pi^2}\ \left((ns)^2 - (ds)^2\right) \tag{5.3.8}$$

and

$$e_1 - e_3 = \frac{1}{4}\ \frac{(q;q^2)_\infty^4 (q^2;q^2)_\infty^4}{(-q^2;q^4)_\infty^4\ (-q;q^2)_\infty^4} = f_3^2(\theta) - f_1^2(\theta) = \frac{K^2}{\pi^2}\ (cs^2 - ds^2) \tag{5.3.9}$$

Proof 5.3.3. Putting $\alpha = \pi$ in (5.3.2) and using the results of Exercise 5.2.3 gives (5.3.4). Similarly (5.3.5) and (5.3.6) follow. Putting $\theta = \pi$ in (5.3.5), $\theta = \pi\tau + \pi$ in (5.3.5) and $\theta = \pi$ in (5.3.6) and using the results of Exercise 5.2.3 again gives first half of (5.3.7) - (5.3.9) respectively. For the remaining identities, simply use (5.3.4) - (5.3.6).

Corollary 5.3.3. [7] From (5.3.1) we have,

$$f(-1, \mathrm{e}^{i\theta})\ f(q, \mathrm{e}^{i\theta}) = \frac{1}{i}\frac{\partial}{\partial\theta}\ f(-q, \mathrm{e}^{i\theta}) + \frac{1}{2}\ f(-q, \mathrm{e}^{i\theta})$$

since, from the definition of $\rho_1(z)$ we have $\rho_1(\mathrm{e}^{i\pi}) = 0$ and $\rho_1(\mathrm{e}^{i\pi\tau}) = \frac{1}{2}$. Thus, on using the results of Exercise 5.2.3 and simplifying,

$$-\mathrm{e}^{\frac{i\theta}{2}} f_1(\theta) f_2(\theta) = \mathrm{e}^{\frac{i\theta}{2}} f_3'(\theta)$$

or,

$$f_3'(\theta) = -f_1(\theta) f_2(\theta).$$

Similarly,

$$f_1'(\theta) = -f_2(\theta) f_3(\theta)$$

and

$$f_2'(\theta) = -f_3(\theta) f_1(\theta).$$

Employing results of Corollary 5.3.2, these reduce to

$$[p'(\theta)]^2 = \prod_{j=1}^{3}(p(\theta) - \mathrm{e}_j).$$

Exercises 5.3.

5.3.1. [7] [Addition Theorems for the Jacobian elliptic functions] Employing (5.3.1) show that

$$\frac{\partial}{\partial\alpha} f(e^{i\alpha}, e^{i\theta}) f(e^{i\beta}, e^{i\theta}) - \frac{\partial}{\partial\beta} f(e^{i\alpha}, e^{i\theta}) f(e^{i\beta}, e^{i\theta})$$
$$= f(e^{i(\alpha+\beta)}, e^{i\theta}) \left(\frac{d}{d\alpha}\rho_1(e^{i\alpha}) - \frac{d}{d\beta}\rho_1(e^{i\beta}) \right)$$

and hence show (on employing the definitions of f_1, f_2, f_3 and p):

$$f_1(\alpha+\beta) = \frac{f_1(\alpha) f_2(\beta) f_3(\beta) - f_1(\beta) f_2(\alpha) f_3(\alpha)}{f_1^2(\beta) - f_1^2(\alpha)}$$

and two similar formulas.

5.4. Cubic Elliptic Function

The following theorem of Ramanujan [13, Second Notebook p.257] provides a cubic analogue of Jacobian elliptic functions, namely the function given by (5.4.3).

Theorem 5.4.1. *[13, p257]*
Let, for $0 < x < 1$,

$$z := {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right),$$
$${}_2F_1(a, b; c; x) := 1 + \sum_1^{\infty} \frac{[a]_n [b]_n}{[c]_n n!}, [a]_n = a(a+1)\cdots(a+n-1), (a)_0 = 1, a \neq 0$$
$$q := e^{-y},$$
$$y := \frac{2\pi}{\sqrt{3}} \frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-x)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; x)}.$$

For $0 \le \phi \le \frac{\pi}{2}$, *define* $\theta = \theta(\phi)$ *by*

$$\theta z = \int_0^{\phi} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \; x\sin^2 t\right) \, \mathrm{d}t, \tag{5.4.1}$$

or, equivalently , by

$$\theta z = \int_0^{\phi} \frac{\cos\left(\frac{1}{3}\sin^{-1}\left(\sqrt{x\sin^2 t}\right)\right)\mathrm{d}t}{\sqrt{1 - x\sin^2 t}}. \tag{5.4.2}$$

Then, for $0 \le \theta \le \frac{\pi}{2}$, *the following inversion holds*

$$\phi = \theta + 3\sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n(1 + 2\cosh(ny))} = \theta + 3\sum_{n=1}^{\infty} \frac{\sin(2n\theta)q^n}{n(1 + q + q^n)}. \tag{5.4.3}$$

The integral and the inverse are clearly analogous to the classical elliptic integral and one of classical Jacobi's elliptic functions.

Proof 5.4.1. Since the proof is protracted, we will be brief. For further details one may see the references [3, 8]. In what follows (small case) z stands for complex number unlike before and the z used hitherto is replaced by (Cap)Z. Define

$$v(z, q) := 1 + 3\sum_{n=1}^{\infty} \frac{(z^n + z^{-n})q^n}{1 + q^n + q^{2n}}, \quad |q| < |z| < |q|^{-1}, \tag{5.4.4}$$

or, globally

$$v(z, q) := 1 + 3\sum_{n=0}^{\infty} \left\{ \frac{zq^{3n+1}}{1 - zq^{3n+1}} - \frac{zq^{3n+2}}{1 - zq^{3n+2}} + \frac{z^{-1}q^{3n+1}}{1 - z^{-1}q^{3n+1}} - \frac{z^{-1}q^{3n+2}}{1 - z^{-1}q^{3n+2}} \right\} \tag{5.4.5}$$

and

$$V(\theta) := v(e^{zi\theta}, q) = 1 + 6\sum_{n=1}^{\infty} \frac{\cos(2n\theta)q^n}{1 + q^n + q^{2n}}. \tag{5.4.6}$$

- Firstly, we can have the following representations (i)-(ii).

(i) v in terms of the cubic theta functions:

$$v\,(e^{i\theta}, q) = \frac{3}{2}\frac{(q;q)_\infty^2 (q^3;q^3)_\infty^2}{(q^2;q^2)_\infty (q^6;q^6)_\infty}\frac{b(q,-e^{i\theta})}{b(q,e^{i\theta})} - \frac{1}{2}\frac{b(q)^2}{b(q^2)} \tag{5.4.7}$$

where $b(q)$ is the one-variable cubic theta function [4,5,6,10] given by

$$b(q) := \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{m^2+mn+n^2} w^{m-n} = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty} \tag{5.4.8}$$

and $b(q,z)$ is the two-variable cubic theta function [4,5,6,10] given by

$$b(q,z) := (q;q)_\infty (q^3;q^3)_\infty \frac{(qz;q)_\infty (qz^{-1};q)_\infty}{(q^3z;q^3)_\infty (q^3z^{-1};q^3)_\infty}, \tag{5.4.9}$$

the other associated cubic theta functions being,

$$
\begin{aligned}
a(q) &:= \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{m^2+mn+n^2} \\
&= 1 + 6\sum_{n=1}^{\infty}\left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}}\right) \\
&= 1 + 6\sum_{n=1}^{\infty}\frac{q^n}{1+q^n+q^{2n}}, \qquad (5.4.10)\\
c(q) &:= \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{(m+\frac{1}{3})^2+(m+\frac{1}{3})(n+\frac{1}{3})+(n+\frac{1}{3})^2} \\
&= 3q^{\frac{1}{3}}\frac{(q^3;q^3)_\infty}{(q;q)_\infty} \qquad (5.4.11)\\
a(q,z) &:= \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{m^2+mn+n^2} z^{m-n} \qquad (5.4.12)\\
b(q,z) &:= \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{m^2+mn+n^2}\omega^{m-n} z^{n} \\
&= (q;q)_\infty(q^3;q^3)_\infty\frac{(qz;q)_\infty(qz^{-1};q)_\infty}{(q^3z;q^3)_\infty(q^3z^{-1};q^3)_\infty} \qquad (5.4.13)\\
c(q,z) &:= \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{(m+\frac{1}{3})^2+(m+\frac{1}{3})(n+\frac{1}{3})+(n+\frac{1}{3})^2} z^{m-n} \\
&= q^{\frac{1}{3}}(q;q)_\infty(q^3;q^3)_\infty(1+z+z^{-1})\frac{(q^3z^3;q^3)_\infty(q^3z^{-3};q^3)_\infty}{(qz;q)_\infty(qz^{-1};q)_\infty} \qquad (5.4.14)
\end{aligned}
$$

(ii)
$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}\theta} &= \frac{\partial}{\partial\theta}v(\mathrm{e}^{i\theta},q) = q(\mathrm{z}-z^{-1}) \qquad (5.4.15)\\
&\times \frac{(z^2q^3;q^3)_\infty(z^{-2}q^3;q^3)_\infty(q;q^3)_\infty(q^2;q^3)_\infty(q^3;q^3)_\infty^4}{(zq;q^3)_\infty^2(z^{-1}q;q^3)_\infty^2(zq^2;q^3)_\infty^2(z^{-1}q^2;q^3)_\infty^2} \qquad (5.4.16)
\end{aligned}
$$

For proofs of (i)-(ii) it is best to study [3] and [8], the proofs given in the latter reference being simpler where the author studies $g_1(\theta,q) = \frac{1}{6}v(\mathrm{e}^{i\theta},q)$ and associated $g_2(\theta,q)$. His proof employs theory of elliptic functions directly and realizes

$v(e^{i\theta}, q)$ to be doubly periodic meromorphic function and hence elliptic.

- Next we define

$$\psi(\theta) := \frac{1}{4x}\left(4 - \frac{V^3(\theta)}{Z^3} - 3\frac{V^2(\theta)}{Z^2}\right) = \frac{1}{4xZ^3}(Z - V)(2Z + V)^2 \tag{5.4.17}$$

so that

$$\frac{d\psi(\theta)}{d\theta} = -\frac{3V}{4xZ^3}(V + 2Z)\frac{dV}{d\theta}. \tag{5.4.18}$$

We now wish to show

$$\left(\frac{d\psi}{d\theta}\right)^2 = 4\psi(1 - \psi)V^2, \tag{5.4.19}$$

or, on using (5.4.17) and (5.4.18),

$$\begin{aligned}\frac{81V^2}{16x^2z^2}&(V + 2Z)^2\left(\frac{dV}{d\theta}\right)^2 \\ &= 4\,\frac{V^2}{4x}\left(4 - \frac{V^3}{Z^3} - 3\frac{V^2}{Z^2}\right)\left(1 - \frac{1}{4x}\left(4 - \frac{V^3}{Z^3} - 3\frac{V^2}{Z^2}\right)\right)\end{aligned}$$

or,

$$81\left(\frac{dV}{d\theta}\right)^2 = (Z - V)(4xZ^3 - (Z - V)(V + 2Z)^2) \tag{5.4.19$'$}$$

or on using (5.4.15),

$$\begin{aligned}81q^2(z-z^{-1})^2(z^2q^3;q^3)_\infty^2(z^{-2}q^3;q^3)_\infty^2(q;q)_\infty^2(q^3;q^3)_\infty^6 \\ = (zq;q^3)_\infty^4(z^{-1}q;q^3)_\infty^4(zq^2;q^3)_\infty^4(z^{-1}q^2;q^3)_\infty^4 \\ \times (Z-V)(4xZ^3-(Z-V)(V+2Z)^2)\end{aligned}$$

or, using (5.4.9),

$$\begin{aligned}81q^2(z-z^{-1})^2(z^2q^3;q^3)_\infty^2(z^{-2}q^3;q^3)_\infty^2(q;q)_\infty^6(q^3;q^3)_\infty^{10} \\ = b^4(q;z)(Z-V)(4xZ^3-(Z-V)(V+2Z)^2).\end{aligned} \tag{5.4.19$''$}$$

For the rest of the proof of (5.4.19), which is protracted, we refer to [3].

- Now it is not difficult to argue, on employing the many properties of $V(\theta)$ proved so far, that

$$0<\psi<1,\quad V>0,\quad \frac{dV}{d\theta}<0,\quad \frac{d\psi}{d\theta}>0,\quad \text{in}\quad 0<\theta<\frac{\pi}{2},\quad \psi(0)=0,\quad \psi(\frac{\pi}{2})=1. \tag{5.4.20}$$

Hence,

$$V(\theta)=\frac{1}{2\sqrt{\psi(\theta)}\sqrt{1-\psi(\theta)}}\frac{d\psi}{d\theta}, 0<\theta<\frac{\pi}{2}$$

or

$$\int_0^\theta V(t)dt=\frac{1}{2}\int_0^\theta \frac{1}{2\sqrt{\psi(t)(1-\psi(t))}}\frac{d\psi}{dt}\cdot dt,\quad 0<\theta<\frac{\pi}{2},$$

or, with

$$V(\theta) =: \frac{d\Phi(\theta)}{d\theta} \tag{5.4.21}$$

and using (5.4.6),

$$\Phi(\theta) = \theta + 6\sum_{1}^{\infty} \frac{\sin(2n\theta)q^n}{n(1+q^n+q^{2n})} = \frac{1}{2}\int_0^{\psi(\theta)} \frac{du}{\sqrt{u(1-u)}} = \sin^{-1}\left(\sqrt{\psi(\theta)}\right) \tag{5.4.22}$$

or, on using (5.4.17),

$$4x\sin^2(\Phi(\theta)) = 4x\psi(\theta) = 4 - \frac{V^3}{Z^3} - 3\frac{V^2}{Z^2}, 0 \le \theta \le \frac{\pi}{2},$$

$$\text{or, on putting } S(x) := \frac{Z}{V}, \tag{5.4.23}$$

$$4(1 - x\sin^2\Phi(\theta))\, S^3(x) - 3S(x) - 1 = 0 \tag{5.4.24}$$

where $S(x)$ is continuous in $(-1, 1)$ and $S(0) = 1$. But,

$$S(x) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; x\sin^2\Phi(\theta)\right)$$
$$= \left(1 - x\sin^2\Phi(\theta)\right)^{-\frac{1}{2}} \cos\left(\frac{1}{3}\sin^{-1}(\sin\Phi(\theta)\sqrt{x})\right)$$

is the unique solution of the cubic (5.4.24) as can be easily verified.

- Thus, we have, on using (5.4.21) and (5.4.23).

$$\frac{Z}{V} = z\frac{d\Theta(\phi)}{d\phi} = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; x\sin^2\phi\right)$$

where $\Theta : \left[0, \frac{\pi}{2}\right] \to \left[0, \frac{\pi}{2}\right]$ is the set theoretic inverse of Φ. Hence,

$$Z\theta = \int_0^{\phi} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; x\sin^2\phi\right) d\phi.$$

This establishes the theorem.

Exercises 5.4.

5.4.1. Show that the integrals on the right sides of (5.4.1) and (5.4.2) are equal.

5.4.2. Obtain (5.4.5) from (5.4.4).

5.4.3. Obtain the form (5.4.19)" of (5.4.19)' in detail.

5.4.4. Prove the various properties of V and ψ given by (5.4.19).

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