LECTURE NOTES

SPECIAL FUNCTIONS AND FUNCTIONS OF MATRIX ARGUMENT: RECENT ADVANCES AND APPLICATIONS IN STOCHASTIC PROCESSES, STATISTICS AND ASTROPHYSICS

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February 2006

SPECIAL FUNCTIONS AND FUNCTIONS OF MATRIX ARGUMENT: RECENT ADVANCES AND APPLICATIONS IN STOCHASTIC PROCESSES, STATISTICS AND ASTROPHYSICS

LECTURE NOTES

of the
4th S.E.R.C. SCHOOL ON SPECIAL FUNCTIONS AND
FUNCTIONS OF MATRIX ARGUMENT:
RECENT ADVANCES AND APPLICATIONS IN
STOCHASTIC PROCESSES, STATISTICS AND ASTROPHYSICS
6th March 2006 to 7th April 2006

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A.M. MATHAI

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This is publication No.33 of the Publications Series of the Centre for Mathematical Sciences (CMS). This is compiled from the lectures to be given at the 4th S.E.R.C. School on Special Functions and Functions of Matrix Argument: Recent Advances and Applications in Stochastic Processes, Statistics and Astrophysics, sponsored by the Department of Science and Technology, Government of India, New Delhi, to be conducted by CMS during the period 6th March to 7th April 2006 and to be hosted by St. Thomas College Pala. The Course Director is Dr. A.M. Mathai and the Course Co-Director is Dr. K.K. Jose.

The Centre would like to thank the Department of Science and Technology, New Delhi, India, for financial assistance which made the School as well as this publication possible and Dr. B.D. Acharya, Advisor to Government of India, Mathematical Sciences Division of DST, New Delhi, for the constant encouragement and whose enthusiasm and initiative are mainly responsible for conducting this School and for bringing out this publication.

Various chapters and sections are compiled from the lecture notes supplied by the various lecturers at the School and the details are given in each chapter. The respective lecturers are solely responsible for the contents and accuracy of their notes.

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PREFACE

The first S.E.R.C.(Science and Engineering Research Council) School on Special Functions was sponsored by DST (Department of Science and Technology), Delhi, and conducted by CMS for six weeks in 1995. The second S.E.R.C. School on Special Functions and Functions of Matrix Argument was sponsored by DST, Delhi, and conducted by CMS for five weeks during the period 29th May to 30th June 2000. In the second School, the main lectures were given by Dr. H.L. Manocha of Delhi, Dr. S. Bhargava of Mysore, Dr. K. Srinivasa Rao and Dr. R. Jagannathan of Chennai and Dr. A.M.Mathai of Montreal, Canada. Supplementary lectures were given and problem sessions were supervised by Dr. K.S.S. Nambooripad and Dr. S.R. Chettiyar of CMS, Dr. R.N. Pillai (retired), Dr. T.S.K. Moothathu and Dr. Yageen Thomas of the University of Kerala and Dr. E. Krishnan of the University College Trivandrum. Lecture Notes were brought out as Publication No.31 of CMS soon after the School was completed. The first two Schools were conducted in Trivandrum (Thiruvananthapuram) area. Dr. A.M. Mathai was the Director and Dr. K.S.S. Nambooripad the Co-Director of these two Schools.

The third S.E.R.C. School on Special Functions and Functions of Matrix Argument: Recent Developments and Recent Applications in Statistics and Astrophysics, sponsored by DST, Delhi, was conducted for five weeks from 14th March to 15th April 2005 by CMS at its Pala Campus, Kerala, India. This time DST, Delhi, wanted the lecture notes to be collected from the main lecturers in advance, compiled and distributed prior to the start of the School. The main lectures for the 3rd School were given by Dr. Hans Haubold of the Outer Space Affairs of the United Nations, Dr. Serge Provost, Professor of Actuarial Sciences and Statistics of the University of Western Ontario, Canada, Dr. R.K. Saxena of Jodhpur, India, Dr. S. Bhargava of Mysore, India and Dr. A.M. Mathai of Montreal, Canada. Supplementary lectures were given by Dr. A. Sukumaran Nair (Chairman, CMS), Dr. K.K. Jose (Director-in-Charge, Pala Campus), Dr. R.N. Pillai, Dr. Yageen Thomas, Dr. V. Seethalakhmi, Dr. Alice Thomas, Dr. E.

Sandhya, Dr. S. Satheesh and Dr. K. Jayakumar. Problem sessions were supervised by Dr. Sebastian George and other lecturers. Extra training in the use of statistical packages, TEX and Mathematica/Maple were given by Joy Jacob, Seemon Thomas and K.M. Kurian of the Department of Statistics of St. Thomas College, Pala. A special lecture sequence on Matlab was conducted by Alexander Haubold of Columbia University, U.S.A. Dr. A.M. Mathai was the Director and Dr.K.K. Jose the Co-Director of this School.

The 4th S.E.R.C. School on Special Functions and Functions of Matrix Argument: Recent Advances and Applications to Stochastic Processes, Statistics and Astrophysics, sponsored by DST, Delhi, will be conducted by CMS for five weeks from 6th March to 7th April 2006 at its Pala Campus. The main lecturers will be Dr. Hans Haubold of the Outer Space Affairs of the United Nations, Dr. P.N. Rathie of the University of Brasilia, Brazil, Dr. S. Bhargava of Mysore, India, Dr. R.K. Saxena of Jodhpur, India and Dr. A.M. Mathai of Montreal, Canada. Supplementary lectures will be given by Dr. A. Sukumaran Nair (Chairman, CMS), Dr. K.S.S. Nambooripad (Director-in-Charge, CMS Trivandrum Campus), Dr. Y Denis of Gorakhpur, India, Dr. N. Unnikrishnan Nair (former Vice-Chancellor of Cochin University of Science and Technology), Dr. Yageen Thomas and Dr. K. Jayakumar. One week will be devoted to Stochastic Processes and recent advances in this area in the 4th School. Problem sessions will be supervised by Dr. Joy Jacob, Dr. Sebastian George and other lecturers. Dr. A.M. Mathai will the Director and Dr.K.K. Jose the Co-Director of this School.

Chapter 1 introduces elementary classical special functions. Gamma, beta, psi, zeta functions, hypergeometric functions and the associated special functions, generalizations to Meijer's G and Fox's H-functions are also examined here. Discussion is confined to basic properties and some applications. Introduction to statistical distribution theory is given here. Some recent extensions of Dirichlet integrals and Dirichlet densities are also given. A glimpse into multivariable special functions such as Appell's functions and Lauricella functions is also given. A special feature of the 4th School will be a discussion of special functions as solutions of differential equations.

Chapter 2 is devoted to stochastic processes, time series models, recent advances in these areas and a discussion on order statistics. Mittag-Leffler processes, alpha-Laplace distribution, infinite divisibility of random variables, geometric infinite divisibility etc will be examined here. Chapter 3 gives a detailed coverage of recent results on Mittag-Leffler functions and fractional calculus. Chapter 4 looks into some applications of special functions in various problems in astrophysics. Current hot topic of Tsallis' statistics will also be examined here. Chapter 5 deals with q-series or basic hypergeometric series. It also gives an insight into Ramanujan's work on theta functions, mock theta function, various types of summation formulae and so on.

Chapter 6 is devoted to an introduction to group theory with emphasis on generalized inverses of matrices and semigroups. Chapter 7 goes into Jacobians of matrix transformations and special functions of matrix argument. Matrix-variate statistical distributions are also dealt with in this chapter. Recent developments on functions of matrix argument and matrix-variate distributions will be given here. Since the lecture notes are compiled in advance, only a summary of what is going to be given in the lectures is given here. More up-to-date materials will be covered in the lectures. In all the S.E.R.C. Schools, conducted under the Directorship of Dr. A.M. Mathai, a serious effort is made so that the participants absorb the materials covered in the School. The classes start at 8.30 am. The first lecture of 08.30 to 10.30 is followed by a problem session from 10.30 to 13.00 hrs on the materials covered in the first lecture. The second lecture of the day will be from 14.00 to 16.00 hrs followed by problem session from 16.00 to 1800 hrs. At the end of every week a written examination is conducted, followed by a personal interview of each participant by the lecturer of that week in the form of an oral examination. Cumulative grades of such weekly examinations appear in the final certificate.

Several People have contributed enormously for the success of this fourth S.E.R.C. School and in making this publication possible. Dr. B.D. Acharya, Advisor to Government of India and Dr. Ashok K. Singh of the Mathematical Sciences Division of DST, New Delhi, are the driving force behind the re-energized mathematical activities in India now. They were kind enough to pursue the matter and get the funds released for running the School as well as for the preparation of this publication. Since the materials for this publication are supplied by various lecturers, there will be some overlaps. Very obvious inconsistencies are removed but some overlapping materials are left there to make the discussions self-contained. Dr. R.K. Saxena, Dr. S. Bhargava, Dr. H.J. Haubold, Dr. P.N. Rathie, Dr. K.K. Jose and Dr. A.M. Mathai are thanked for making their notes available in advance for this publication. The material was typeset at CMS office by K.H. Soby, Dr. Joy Jacob, Seemon Thomas, Dr. Sebastian George, Dr. K.K. Jose and myself. For making the indexes Jaisymol Thomas and Ashly P. Jose spent a lot of time. CMS would like to thank K.H. Soby, Jaisymol Thomas and Ashly P. Jose for their sincere and dedicated work.

CMS would like to express its sincere thanks for the main lecturers Dr. Hans Haubold, Dr. R.K. Saxena, Dr. S. Bhargava, Dr. K.K. Jose, Dr. A.M. Mathai and Dr. P.N. Rathie, and the supplementary lecturers Dr. A. Sukumaran Nair, Dr. K.S.S. Nambooripad, Dr. N. Unnikrishnan, Dr. Y. Denis, Dr. Yageen Thomas, Dr. K. Jayakumar and the problem session leaders Dr. Sebastian George and Dr. Joy Jacob. CMS would like to thank Barbara Haubold and Dr. P.N. Rathie for sparing their time for associated lectures for the general public and Dr. Joy Jacob, Seemon Thomas and K.M. Kurian for running special training programs on the use of statistical packages, TEX and Mathematica/Maple.

We express our sincere gratitude to the Management and Principal, St. Thomas College, Pala for providing all facilities in the College for the successful conduct of the School. The enthusiasm and co-operation shown by Rev. Dr. M.M. Mathew, Principal and Rev. Dr. Mathew John K, Bursar, St. Thomas College, Pala, have contributed significantly for the success of the School. CMS would like to express its gratitude to the College canteen authorities for supplying lunch, coffee/tea to the participants. The faculty and students from the Departments of Mathematics, Statistics and Physics at St. Thomas College Pala did a lot of voluntary work. CMS would like to thank each and every one who directly or indirectly helped to run the School and make it a success.

Pala, Kerala, India	Dr. A.M. Mathai
15th February, 2006	Director of the School and Director of CMS

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GLOSSARY OF SYMBOLS

$(b)_r$	Pochammer Symbol	Notation 1.0.1	1
$n!$	factorial n or n factorial	Notation 1.0.2	2
$\binom{n}{r}$	number of combinations of n taken r at a time	Notation 1.0.3	2
$R(\cdot)$	real part of $(\cdot)$	Notation 1.1.1	3
$\Gamma(z)$	gamma function	Notation 1.1.1	3
γ	Euler's constant	Notation 1.1.5	4
π	pi, mathematical constant	Definition 1.1.6	4
$\wedge$	wedge or skew symmetric product	Notation 1.1.3	7
J	Jacobian	Definition 1.1.7	8
$\psi(\cdot)$	psi function	Section 1.2	12
$\zeta(\rho)$	Riemann zeta function	Section 1.2.1	13
$\zeta(\rho, a)$	generalized zeta function	Section 1.2.1	13
$M_f(s)$	Mellin transform of f	Section 1.3	15
$L_f(t)$	Laplace transform of f	Section 1.3	18
$F_f(t)$	Fourier transform of f	Section 1.3	19
$E(\cdot)$	expected value	Notation 1.5.1	31
$\mathrm{Var}(\cdot)$	variance of $(\cdot)$	Definition 1.5.2	32
$\mathrm{Cov}(\cdot)$	covariance	Exercises 1.5.7	38
$B(\alpha, \beta)$	beta function	Section 1.6	39
${}_pF_q(\cdot)$	hypergeometric function	Section 1.7	48
$\gamma(\alpha; b)$	incomplete gamma function	Section 1.7	49
$\Gamma(\alpha; b)$	incomplete gamma function	Section 1.7	50
$b(\alpha, \beta; t)$	incomplete beta function	Section 1.7	50
$B(\alpha, \beta; t)$	incomplete beta function	Section 1.7	51
$G_{p,q}^{m,n}[\cdot]$	G-function	Section 1.8	54
$J_\nu(\cdot)$	Bessel function	Section 1.8	60
$I_\nu(\cdot)$	Bessel function	Section 1.8	60
$H_{p,q}^{m,n}(\cdot)$	H-function	Section 1.9	61
$E_{\alpha,\beta}(\cdot)$	Mittag-Leffler function	Section 1.9	64
$f_A(\cdot), f_B(\cdot),$	Lauricella function	Section 1.10	65
$f_C(\cdot), f_D(\cdot)$	Lauricella function	Section 1.10	65

$H_\nu^{(1)}(x), H_\nu^{(2)}(x)$	Bessel function of the third kind	Section 1.11.8	85
$L_n^{(\alpha)}(x)$	Laguerr polynomial	Section 1.11.9	86
$\{X(t,w); t \in T; w \in \Omega\}$	stochastic process	Section 2.1	95
$TEAR(1)$	tractable exponential Autoregressive Process	Section 2.4	129
$TMLAR(1)$	Tractable Mittag-Leffler Autoregressive Process	Section 2.4.1	129
$TSMLAR(1)$	Tractable Semi-Mittag-Leffler Autoregressive Process	Section 2.4.1	129
$NEAR(1)$	New Exponential Autoregressive Process	Section 2.4.2	131
$EAR(1)$	Exponential Autoregressive Process	Section 2.4.2	132
$NMLAR(1)$	New Mittag-Leffler Autoregressive Process	Section 2.4.3	132
$NSMLAR(1)$	New Semi-Mittag-Leffler Autoregressive Process	Section 2.4.4	134
$MLTAR(1)$	Mittag-Leffler Tailed Autoregressive Process	Section 2.5.2	138
$SMLTAR(1)$	Semi-Mittag-Leffler Tailed Autoregressive Process	Section 2.5.2	140
$MOSW(1)$	Marshall-Olkin Semi-Weibull Process	Section 2.6.2	144
$MOSP(1)$	Marshall-Olkin Semi-Pareto Process	Section 2.6.2	145
$MOGW(1)$	Marshall-Olkin Generalized Weibull Process	Section 2.6.4	146
$AR(1)$	first order Autoregressive Process	Section 2.6.3	147
$Y_{[r:n]}$	r^{th} order statistic from a sample of n observations	Section 2.7	159
$BLUE$	Best Linear Unbiased Estimator	Section 2.7.1	159

$F(s) = L\{f(t); s\}$	Laplace transform	Notation 3.1.1	175
$L^{-1}\{f(s); t\}$	inverse Laplace transform	Notation 3.1.2	175
${}_0I_x^{-\nu}$	fractional integral	Section 3.1.2	177
${}_0D_x^{\alpha}$	fractional derivative	Section 3.1.3	177
${}^{C}_{0}\,{}_{a}D_x^{\alpha}$	Caputo derivative	Section 3.1.4	178
$E_\alpha(\cdot)$	Mittag-Leffler function	Exercises 3.1	179
$m\{f(x); s\}$	Mellin transform	Section 3.2.1	180
$m^{-1}\{f^*(s); x\}$	inverse Mellin transform	Section 3.2.1	180
$\mathbb{I}[f(x)],$	Kober operator of the first kind	Notation 3.3.1	183
$\mathbb{I}[\alpha, \eta : f(x)]$	Kober operator of the first kind	Notation 3.3.1	183
$\mathbb{I}(\alpha, \eta)f(x)E_{0,x}^{\alpha,\eta}\ f$	Kober operator of the first kind	Notation 3.3.1	183
$\mathbb{I}_x^{n,\alpha}\ f$	Kober operator of the first kind	Notation 3.3.1	183
$\mathbb{R}[f(x)]$	Kober operator of the second kind	Notation 3.3.2	183
$\mathbb{R}[\alpha, \zeta : f(x)]$	Kober operator of the second kind	Notation 3.3.2	183
$\mathbb{R}(\alpha, \zeta)f(x)$	Kober operator of the second kind	Notation 3.3.2	183
$K_{x,\infty}^{\alpha,\zeta}\ f, K_x^{\zeta,\alpha}\ f$	Kober operator of the second kind	Notation 3.3.2	183
$\mathbb{I}[\cdot], \mathbb{K}[\cdot]$	generalized Kober operators	Section 3.4.	186
$I_{0,x}^{\alpha,\beta,\eta}(\cdot)$	fractional integral / derivative of the first kind	Definition 3.4.7	189
$J_{x,\infty}^{\alpha,\beta,\eta}(\cdot)$	fractional integral / derivative of the first kind	Definition 3.4.8	189
$E_\alpha(x)$	Mittag-Leffler function	Notation 3.5.1	195
$E_{\alpha,\beta}(x)$	generalized Mittag-Leffler function	Notation 3.5.2	195
$I_{0+}^{\alpha}f$	Riemann-Liouville left-sided integral	Notation 3.5.3	195
$I_{-}^{\alpha}f$	Riemann-Liouville right-sided integral	Notation 3.5.4	195
$D_{0+}^{\alpha}f$	Riemann-Liouville left-sided derivative	Notation 3.5.5	195
$D_{-}^{\alpha}f$	Riemann-Liouville right-sided derivative	Notation 3.5.6	195
$E_{\alpha,\beta}^{\gamma}(x)$	generalized Mittag-Leffler function (Prabhakar)	Notation 3.5.7	195
$R_\odot$	solar radius	Section 4.1	213
$M(r)$	solar mass at r	Section 4.1	214

$M_\odot$	solar mass	Section 4.1	214
$\rho(r)$	solar density at r	Section 4.1	214
$P(r)$	pressure at r	Section 4.1	214
$T(r)$	temperature at r	Section 4.1	215
$L(r)$	luminosity at r	Section 4.1	215
$(a;q)_\infty$	infinite product	Section 5.2	231
$(a;q)_n$	ratio of infinite product	Section 5.2	231
$\Bbbk$	field	Section 6.1.1	250
$A_1 \oplus A_2$	symmetric difference	Example 6.1.1	250
$\cup$	union of sets	Example 6.1.1	250
$\cap$	intersection of sets	Example 6.1.1	250
$\mathbb{C}$	complex field	Example 6.1.2	251
$\mathbb{C}^{(n)}$	complex field	Example 6.1.3	251
$\supset, \supseteq$	contains	Section 6.1.1	252
$\subseteq, \subset$	contained in	Section 6.1.2	253
$\mathcal{L}\langle X\rangle$	Linear span of X	Section 6.1.2	254
$\dim V$	dimension of V	Section 6.1.2	257
$\langle u, v\rangle$	inner product	Section 6.1.3	258
$\lVert(\cdot)\rVert$	norm of $(\cdot)$	Section 6.1.3	259
$x \perp y$	orthogonality	Section 6.1.3	259
$X^\perp$	orthogonal complement	Section 6.1.3	259
$\mathsf{M}at_{n\times m}$	$m \times n$ matrix	Section 6.2.2	268
$\mathrm{tr}(X)$	trace of X	Section 7.0	289
X'	transpose of X	Section 7.0	289
$\mathrm{d}x \wedge \mathrm{d}y$	wedge product	Section 7.0	289
$\lvert(.)\rvert$	determinant of $(\cdot)$	Section 7.1	290
J	Jacobian	Section 7.1	290
$\Gamma_p(\alpha)$	real matrix-variate gamma	Definition 7.2.1	297
$B_p(\alpha,\beta)$	real matrix-variate beta	Definition 7.2.2	298
$L_f(t_1,...t_k)$	Laplace transform	Section 7.3	302
$L_f(T^*)$	Laplace transform	Definition 7.3.1	303
$C_K(Z)$	zonal polynomial	Section 7.4.2	310
$\Gamma_p(\alpha,\kappa)$	generalized gamma	Section 7.4.2	311
$M_\alpha(f)$	M transform	Section 7.4.3	313