CHAPTER 1

ELEMENTARY SPECIAL FUNCTIONS

[*This chapter is based on the lectures of Professor A.M. Mathai of McGill University, Canada (Director of the 5^{th} SERC School).*]

1.0. Introduction

Some preliminaries of special functions and statistical distributions are given here. Details are available from the following sources, which are accessible to the participants of the 5^{th} SERC School:

(1) *Notes of the 2^{nd} SERC School*. (Publication No 31 of the Centre for Mathematical Sciences (CMS)), 2000.
(2) *Notes of the 3^{rd} SERC School*. (Publication No 32 of the Centre for Mathematical Sciences (CMS)), 2005.
(3) *Notes of the 4^{th} SERC School*. (Publication No 33 of the Centre for Mathematical Sciences (CMS)), 2006.
(4) Mathai, A.M. (1993). *"A Handbook of Generalized Special Functions for Statistical and Physical Sciences"*, Oxford University Press, Oxford, U.K.
(5) Mathai, A.M. and Saxena, R.K. (1978). *"The H-Function with Applications in Statistics and Other Disciplines"*, Wiley Halsted, New York.
(6) Mathai, A.M. and Saxena, R.K. (1973). *"Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences"*, Lecture Notes No 348, Springer-Verlag, Heidelberg, Germany.

Notation 1.0.1. Pochammer symbol

$$(a)_m = a(a+1)\cdots(a+m-1),\ (a)_0 = 1,\ a \neq 0 \tag{1.0.1}$$

For example,

$$\left(-\frac{2}{3}\right)_2 = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}+1\right) = \left(-\frac{2}{9}\right); \qquad (-3)_3 = (-3)(-2)(-1) = -6;$$

$$(-3)_5 = (-3)(-2)(-1)(0)(1) = 0; \qquad \left(\frac{1}{2}\right)_3 = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right) = \frac{15}{8}.$$

Notation 1.0.2. **Factorial n or n factorial**

$$n! = (1)(2)\cdots(n),\ 0! = 1 \text{ (convention)}. \tag{1.0.2}$$

For example,

$$3! = (1)(2)(3) = 6; \qquad \frac{2}{3}! = \text{not defined}$$

$$(-2)! = \text{not defined}; \qquad 1! = 1;\ \ 0! = 1 \quad \text{(convention)}.$$

Notation 1.0.3. **Number of combinations of n taken r at a time**

$$\binom{n}{r} = \text{number of subjects of r distinct objects from a set of } n \tag{1.0.3}$$

$$= \frac{n(n-1)\cdots(n-(r-1))}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!},\quad 0 \le r \le n.$$

For example,

$$\binom{3}{1} = \frac{3}{1!} = 3;\ \binom{3}{2} = \frac{(3)(2)}{2!} = 3;\ \binom{n}{1} = \frac{n}{1!} = n;$$

$$\binom{n}{n-1} = \frac{n(n-1)\cdots(n-(n-1))}{(n-1)!} = n \Longrightarrow \binom{n}{1} = \binom{n}{n-1};$$

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1;\ \binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = 1 \Longrightarrow \binom{n}{0} = \binom{n}{n};$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)![n-(n-r)]!} = \binom{n}{n-r};\ \binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1};$$

$$\binom{-2}{3} = \text{ not defined as a combination;} \binom{1/3}{2} = \text{not defined as a number of combinations.}$$

But if $\binom{n}{r}$ is not treated as a number of combinations but defined in terms of Pochammer symbol as

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{(-1)^r(-n)(-n+1)\cdots(-n+r-1)}{r!} \qquad (1.0.4)$$
$$= \frac{(-1)^r(-n)_r}{r!}$$

then

$$\binom{-2/3}{2} = \frac{(-1)^2}{2!}(\frac{2}{3})\Big(\frac{2}{3}+1\Big) = \frac{5}{9};\ \binom{1/2}{3} = \frac{(-1)^3}{3!}\Big(\frac{-1}{2}\Big)\Big(\frac{-1}{2}+1\Big)\Big(\frac{-1}{2}+2\Big) = \frac{1}{16}.$$

note that

$$(a)_{m+n} = (a)_m(a+m)_n = (a)_n(a+n)_m \qquad (1.0.5)$$

1.1. Gamma and Related Functions

Notation 1.1.1. $\Gamma(z) =$ **gamma** z

A gamma function is defined in many ways. Some of the definitions, along with the necessary conditions are the following:

Definition 1.1.1.

$$\Gamma(z) \ = \ \lim_{n\to\infty} \frac{n!\ n^z}{z(z+1)\cdots(z+n)},\quad z \neq 0, -1, -2, \cdots \qquad (1.1.1)$$

Definition 1.1.2.

$$\Gamma(z) \ = \ z^{-1}\prod_{n=1}^{\infty}\Big(1+\frac{1}{n}\Big)^z\Big(1+\frac{z}{n}\Big)^{-1}. \qquad (1.1.2)$$

Definition 1.1.3.

$$\frac{1}{\Gamma(z)} \ = \ z\lim_{n\to\infty}\left\{n^{-z}\prod_{k=1}^{n}\Big(1+\frac{z}{k}\Big)\right\}. \qquad (1.1.3)$$

Definition 1.1.4.

$$\frac{1}{\Gamma(z)} \ = \ ze^{\gamma z}\prod_{n=1}^{\infty}\left[\Big(1+\frac{z}{n}\Big)e^{-z/n}\right] \qquad (1.1.4)$$

where γ is the Euler's constant.

Notation 1.1.2.

$$\gamma = \lim_{n\to\infty}\{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\} \approx 0.5772156649015328606065 12. \quad (1.1.5)$$

Definition 1.1.5.

$$\Gamma(z) \;=\; p^z \int_0^\infty t^{z-1}\mathrm{e}^{-\mathrm{pt}}\mathrm{dt}, \Re(\mathrm{p}) > 0, \Re(\mathrm{z}) > 0 \quad (1.1.6)$$

where $\Re(.)$ denotes the real part of (.).

Definition 1.1.6.

$$\frac{1}{\Gamma(z)} \;=\; \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-z}e^t dt,\; c > 0, \Re(z) > 0,\; i = \sqrt{-1} \quad (1.1.7)$$

where π is the mathematical constant,

$$\pi \approx 3.14159265358979323846 2643.$$

Thus, from the Laplace representation in (1.1.6) we have an integral representation

$$\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\mathrm{d}t,\; \Re(z) > 0. \quad (1.1.8)$$

In general, $\Gamma(z)$ exists for all values of z, positive or negative, except at the points $z = 0, -1, -2, \cdots$. These are the poles of $\Gamma(z)$. But for the integral representation in (1.1.8) to hold the real part of z must be positive. Thus, for example,

$\Gamma(5)$ exists; $\Gamma(-\frac{1}{2})$ exists; $\Gamma(0)$ does not exist; $\Gamma(-3)$ does not exist.
It is not difficult to show that

$$\Gamma(z) = (z-1)\Gamma(z-1) = (z-1)(z-2)\cdots(z-r)\Gamma(z-r) \quad (1.1.9)$$

when $\Gamma(z)$ and $\Gamma(z-r)$ are defined. It is easily established from the integral representation in (1.1.8) by integrating by parts. The property holds for other definitions also. Thus, for example,

$$\begin{aligned}\Gamma(n) &= (n-1)(n-2)\cdots 1\Gamma(1) \text{ but } \Gamma(1)=1\\ &= (n-1)! \text{ for } n=1,2,\cdots \qquad (1.1.10)\end{aligned}$$

$$\Gamma\left(\frac{1}{2}\right) = \left(\frac{1}{2}-1\right)\Gamma\left(\frac{1}{2}-1\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right) \Rightarrow \Gamma\left(-\frac{1}{2}\right) = -2\Gamma\left(\frac{1}{2}\right).$$

$$\Gamma\left(\frac{7}{2}\right) = \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\Gamma\left(\frac{1}{2}\right).$$

By using the property in (1.1.9) we can reduce any gamma function

$$\Gamma(z) = (\text{a few factors})\Gamma(\alpha), 0 < \alpha \le 1 \qquad (1.1.11)$$

and $\Gamma(\alpha)$ for $0 < \alpha \le 1$ is extensively tabulated. For computational purposes one can use (1.1.11) and the extensive numerical tables for $\Gamma(\alpha), 0 < \alpha \le 1$. It can also be shown that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. The proof is simple in terms of the integral representations in (1.1.8). Consider

$$\begin{aligned}\left[\Gamma(\frac{1}{2})\right]^2 = \Gamma(\frac{1}{2})\Gamma(\frac{1}{2}) &= \left[\int_0^\infty x^{\frac{1}{2}-1}e^{-x}dx\right]\left[\int_0^\infty y^{\frac{1}{2}-1}e^{-y}dy\right]\\ &= \int_0^\infty\int_0^\infty x^{\frac{1}{2}-1}y^{\frac{1}{2}-1}e^{-(x+y)}dxdy.\end{aligned}$$

Put $x = r\cos^2\theta, y = r\sin^2\theta, 0 \le r < \infty, 0 \le \theta \le \frac{\pi}{2}$, the Jacobian is $2r\sin\theta\cos\theta$, integrate out r and θ to see that the right side reduces to π, and hence the result. Therefore

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \qquad (1.1.12)$$

Also

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \qquad (1.1.13)$$

when the gammas are defined.

1.1.1. Multiplication formula for a gamma function

$$\Gamma(mz) = (2\pi)^{\frac{1-m}{2}} m^{mz-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)\cdots\Gamma\left(z+\frac{m-1}{m}\right),\ m=1,2,\cdots \qquad (1.1.14)$$

For example,

$$\Gamma(2z) = (2\pi)^{\frac{1-2}{2}} 2^{2z-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \pi^{-\frac{1}{2}}z^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

is known as the *duplication formula* for gamma functions. For example,

$$1 = \Gamma(1) = \Gamma\left[2\left(\frac{1}{2}\right)\right] = \pi^{-\frac{1}{2}}2^{1-1}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\right) = \pi^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)\Gamma(1) \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

$$1 = \Gamma\left[3\left(\frac{1}{3}\right)\right] = (2\pi)^{\frac{1-3}{2}}3^{1-\frac{1}{2}}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma(1) \Rightarrow \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}.$$

By using the infinite product definitions for trigonometric functions we can establish the following results:

$$\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec} \pi z; \tag{1.1.15}$$

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z}\operatorname{cosec} \pi z; \tag{1.1.16}$$

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right) = \pi \sec \pi z. \tag{1.1.17}$$

Exercises 1.1.

1.1.1. Evaluate the following whenever they exist.

(a) $\left(-\frac{2}{3}\right)_3$; (b) $(-3)_4$; (c) $(1)_n$; (d) $(0)_2$.

1.1.2. Evaluate the following, interpreting as the number of combinations, whenever they exist.

(a) $\binom{2/3}{5}$; (b) $\binom{-2}{3}$; (c) $\binom{2}{3}$; (d) $\binom{4}{2}$; (e) $\binom{100}{4}$.

1.1.3. An M.Sc Mathematics class has 5 boys and 9 girls. A committee of 4 persons is to be chosen, consisting of 2 boys and 2 girls. (a) How many total choices are there? (b) How many choices are there if there is no restriction on the number of boys and girls in the committee?

1.1.4. Prove that Definitions 1.1.3 and 1.1.4 are one and the same.

1.1.5. Evaluate the following in terms of $\Gamma(\alpha), 0 < \alpha \leq 1$.

(a) $\Gamma(-\frac{5}{2})$; (b) $\Gamma(-\frac{3}{4})$; (c) $\Gamma(\frac{7}{2})$; (d) $\Gamma(8)$.

1.1.6. Evaluate the following:

(a) $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})$; (b) $\Gamma(\frac{1}{6})\Gamma(\frac{2}{3})$.

1.1.7. Show that $\Gamma(\frac{1}{6})\Gamma(\frac{5}{6}) = 2\pi$.

1.1.8. Show that $z\Gamma(z) = \Gamma(z+1)$ by using Definition 1.1.1.

1.1.9. Show that $z\Gamma(z) = \Gamma(z+1)$ by using Definition 1.1.2.

1.1.10. Show that $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-z}\left(1+\frac{z}{n}\right) = \lim_{n\to\infty} n^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)$.

1.2. Bernoulli Polynomials

Notation 1.2.1.

$$B_k^{(a)}(x) \ : \ \text{generalized Bernoulli polynomial of order } k$$
$$B_k^{(1)}(x) = B_k(x) : \ \text{Bernoulli polynomial of order } k$$
$$B_k(0) = B_k : \ \text{Bernoulli number of order } k$$

Definition 1.2.1.

$$\frac{t^a e^{xt}}{(e^t-1)^a} = \sum_{k=0}^{\infty}\frac{t^k}{k!}B_k^{(a)}(x), \ |t| < 2\pi; \tag{1.2.1}$$

$$\frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty}\frac{t^k}{k!}B_k(x), \ |t| < 2\pi; \tag{1.2.2}$$

$$\frac{t}{e^t-1} = \sum_{k=0}^{\infty}\frac{t^k}{k!}B_k, \ |t| < 2\pi. \tag{1.2.3}$$

1.2.1. Some basic properties

$$
\begin{aligned}
&B_k^{(0)}(x) = x^k;\\
&B_0^{(a)}(x) = 1;\\
&B_k^{(a)}(x) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left\{\mathrm{e}^{xt}\left[\frac{t^a}{(\mathrm{e}^t-1)^a}\right]\right\} \text{ at } t=0.
\end{aligned} \tag{1.2.4}
$$

For computational purposes we need the first few Bernoulli polynomials. These will be listed here.

1.2.2. The first three generalized Bernoulli polynomials

$$B_0^{(a)}(x) = 1;\ B_1^{(a)}(x) = x - \frac{a}{2};\ B_2^{(a)}(x) = x^2 - ax + \frac{a(3a-1)}{12}.$$

From here one has the Bernoulli polynomials and Bernoulli numbers:

$$
\begin{aligned}
&B_0(x) = 1,\ B_1(x) = x - \frac{1}{2},\ B_2(x) = x^2 - x + \frac{1}{6}\\
&B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6}.
\end{aligned}
$$

1.2.3. Asymptotic expansions of gamma functions

Several types of asymptotic results are available on gamma functions. The ones which are useful for computational purposes are given below.

$$
\begin{aligned}
\ln\Gamma(z) &= \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi)\\
&\quad + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k-1)(2k)z^{2k-1}} + O(z^{-2n-1})
\end{aligned} \tag{1.2.5}
$$

for $|\arg z| \le \pi - \epsilon,\ \epsilon > 0$;

$$
\begin{aligned}
\Gamma(z) &= (2\pi)^{\frac{1}{2}} z^{z-\frac{1}{2}}\mathrm{e}^{-z}\\
&\quad \times \exp\left\{1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} + O(z^{-4})\right\}
\end{aligned} \tag{1.2.6}
$$

for $|\arg z| \leq \pi - \epsilon,\ \epsilon > 0$. Here (1.2.6) is an explicit form of (1.2.5) and the leading term in (1.2.6) is the famous *Stirling's formula*.

$$\ln\Gamma(z+a) = \frac{1}{2}\ln(2\pi) + \left(z + a - \frac{1}{2}\right)\ln z - z + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}B_{k+1}(a)}{k(k+1)z^k} \tag{1.2.7}$$

for $|\arg(z+a)| \leq \pi - \epsilon,\ \epsilon > 0$ and a bounded.

Exercises 1.2.

1.2.1 Prove the duplication formula for gamma functions.

1.2.2 Prove the multiplication formula for gamma functions.

1.2.3 Give another derivation, other than the one given above, to show that $\Gamma(1/2) = \sqrt{\pi}$.

1.2.4 From the definition of the generating function evaluate the first 4 generalized Bernoulli polynomials of order k.

1.2.5 Evaluate $\frac{\Gamma(z+a)}{\Gamma(z+b)}$ if $b - a = m,\ m = 1, 2, \ldots$

1.3. The Psi and Zeta Functions

The logarithmic derivative of a gamma function is the psi function and successive derivatives give generalized zeta functions.

Notation 1.3.1. $\psi(z)$**: psi z**

Definition 1.3.1.

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{dz}}\ln\Gamma(z) = \frac{1}{\Gamma(z)}\frac{\mathrm{d}}{\mathrm{dz}}\Gamma(z) \tag{1.3.1}$$

$$\ln\Gamma(z) = \int_1^z \psi(x)\mathrm{d}x.$$

By taking logarithm and then differentiating one can obtain many properties for psi functions from the corresponding properties of gamma functions. For example, from (1.1.9.) we have

$$\psi(z) = \frac{1}{z-1} + \frac{1}{z-2} + \cdots + \frac{1}{z-r} + \psi(z-r). \tag{1.3.2}$$

The following are some further properties :

$$\psi(z) = -\gamma - \frac{1}{z} + z\sum_{k=1}^{\infty} \frac{1}{k(z+k)} \tag{1.3.3}$$

$$\psi(z) = -\gamma + (z-1)\sum_{k=0}^{\infty} \frac{1}{(k+1)(z+k)} \tag{1.3.4}$$

$$\psi(1) = -\gamma \tag{1.3.5}$$

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2 \tag{1.3.6}$$

$$\psi(z) - \psi(1-z) = -\pi \cot \pi z \tag{1.3.7}$$

$$\psi\left(\frac{1}{2}+z\right) - \psi\left(\frac{1}{2}-z\right) = \pi \tan \pi z \tag{1.3.8}$$

where γ is the Euler's constant.

1.3.1. Generalized zeta function

Notation 1.3.2.

$\zeta(\rho, a)$: generalized zeta function
$\zeta(\rho)$: Riemann zeta function

Definition 1.3.2.

$$\zeta(\rho, a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^{\rho}},\ \Re(\rho) > 1,\ a \neq 0, -1, -2, \cdots \tag{1.3.9}$$

$$\zeta(\rho) = \sum_{k=1}^{\infty} \frac{1}{k^{\rho}},\ \Re(\rho) > 1\ . \tag{1.3.10}$$

For $\rho \leq 1$ the series is divergent. Successive derivatives of (1.3.4) yield the following results:

$$\frac{d^2}{dz^2}\ln\Gamma(z) = \frac{d}{dz}\psi(z) = \sum_{k=0}^{\infty}\frac{1}{(z+k)^2} = \zeta(2,z) \tag{1.3.11}$$

$$\begin{aligned}\frac{d^r}{dz^r}\ln\Gamma(z) &= \frac{d^{r-1}}{dz^{r-1}}\psi(z) = \begin{cases}\psi(z), & \text{for } r=1\\ (-1)^r(r-1)!\zeta(r,z), & \text{for } r\geq 2\end{cases}\\ &= (-1)^r(r-1)!\sum_{k=0}^{\infty}\frac{1}{(z+k)^r}.\end{aligned} \tag{1.3.12}$$

Explicit evaluations can be done in a few cases.

$$\zeta(2) = \zeta(2,1) = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6} \tag{1.3.13}$$

$$\zeta(4) = \zeta(4,1) = \sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90} \tag{1.3.14}$$

$$\zeta(2n) = \zeta(2n,1) = \sum_{k=1}^{\infty}\frac{1}{k^{2r}} = \frac{(-1)^{r+1}(2\pi)^{2r}}{2(2r)!}B_{2r} \tag{1.3.15}$$

where B_{2r} is a Bernoulli number. For these and other results see Mathai (1993).

Exercises 1.3.

1.3.1. Prove formula (1.3.4) by using (1.1.9).

1.3.2. Prove formula (1.3.3).

1.3.3. Prove formula (1.3.6) by using the duplication formula for gamma functions.

1.3.4. Show that

$$\psi(1+n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma\ .$$

1.3.5. Evaluate $\psi(-\frac{3}{2})$.

1.3.6. Evaluate $\psi(5)$.

1.3.7. If $\ln\Gamma(z+1) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$ evaluate $a_n,\ \ n = 0,1,2,\cdots$

1.3.8. Show that $\zeta(k, \frac{1}{2}) = (2^k - 1)\zeta(k)$.

1.3.9. show that $\zeta(k, -\frac{3}{2}) = (-1)^k\left(2^k\right)\left[1 + \frac{1}{3^k}\right] + \zeta(k, \frac{1}{2})$.

1.3.10. Show that

$$\zeta\left(k, z - \frac{2r+1}{2}\right) = \frac{1}{\left(z - \frac{1}{2}\right)^k} + \cdots + \frac{1}{\left(z - \frac{2r+1}{2}\right)^k} + \zeta\left(k, z + \frac{1}{2}\right), r = 0, 1, \cdots, \quad k = 2, 3, \cdots$$

1.4. Essentials of Statistical Distribution Theory

The mathematical aspects of statistical distributions will be defined and discussed here. We will not be dealing with random variables, discrete probability functions, mixed situations etc here. Hence density functions defined on a continuum of points in the real case will be considered here.

Let $f(x_1, \cdots, x_k)$ be a non-negative integrable scalar function with the total integral unity in the real scalar variables $x_1, \cdots, x_k$.

Definition 1.4.1. A density function.

If $f(x_1, \cdots, x_k)$ satisfies the following conditions:

(i) $f(x_1, \cdots, x_k) \geq 0$ for all $x_1, \cdots, x_k$,
(ii) $\int_{x_1}, \cdots, \int_{x_k} f(x_1, \cdots, x_k) \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k = 1$

then $f(x_1, \cdots, x_k)$ is called a joint density function of the real scalar random variables $x_1, \cdots, x_k$ where $\wedge$ denotes the wedge product or skew symmetric product of differentials.

Example 1.4.1. Check whether the following are density functions:

(1) $f_1(x) = \frac{1}{\theta}, 0 \leq x \leq \theta$ and $f_1(x) = 0$ elsewhere;

(2) $f_2(x) = \frac{1}{x}, 1 \le x \le \infty$ and $f_2(x) = 0$ elsewhere;

(3) $f_3(x) = \begin{cases} x, \ 0 \le x \le 1 \\ 2-x, 1 \le x \le 2 \end{cases}$ and zero elsewhere;

(4) $f_4(x_1, x_2) = 2, 0 \le x_1 \le x_2 \le 1$ and zero elsewhere;

(5) $f_5(x_1, \cdots, x_k) = \mathrm{e}^{-(x_1+\cdots+x_k)}, 0 \le x_j < \infty, j = 1, \cdots, k$ and zero elsewhere.

Solutions 1.4.1. Obviously the functions in (1) to (5) are non-negative and hence we need to check the second condition only.

(1)
$$\int_{-\infty}^{\infty} f_1(x)\mathrm{d}x = 0 + \int_0^{\theta} \frac{1}{\theta}\mathrm{d}x = \left[\frac{x}{\theta}\right]_0^{\theta} = 1.$$
Hence $f_1(x)$ is a density. This is known as a *uniform density* or the random variable x is said to be uniformly distributed over the closed interval $[0, \theta], \theta > 0$, where θ is an unknown constant. Unknown constants in a density are called *parameters*. Hence θ is a parameter here.

(2)
$$\int_{-\infty}^{\infty} f_2(x)\mathrm{d}x = 0 + \int_1^{\infty} \frac{1}{x}\mathrm{d}x = \Big[\ln x\Big]_1^{\infty} = \infty.$$
The integral does not converge to 1. Hence $f_2(x)$ is not a density.

(3)
$$\int_{-\infty}^{\infty} f_3(x)\mathrm{d}x = 0 + \int_0^1 x\mathrm{d}x + \int_1^2 (2-x)\mathrm{d}x = \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 = 1.$$
Hence $f_3(x)$ is a density function.

(4)
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_4(x_1, x_2)\mathrm{d}x_1 \wedge \mathrm{d}x_2 = 0 + \int_{x_2=0}^{1}\Big[\int_{x_1=0}^{x_2} 2\mathrm{d}x_1\Big]\mathrm{d}x_2 = \int_{x_2=0}^{1} 2x_2\mathrm{d}x_2 = 1.$$
Note that the region of integration is either

$$\{(x_1, x_2)|0 \le x_1 \le x_2 \text{ and } 0 \le x_2 \le 1\} \text{ or } \{(x_1, x_2)|x_1 \le x_2 \le 1 \text{ and } 0 \le x_1 \le 1\}.$$

We may use either of these. Thus, $f_4(x_1,x_2)$ is a joint density function of the real scalar random variables x_1 and x_2.

(5)

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_5(x_1,\cdots,x_k)\mathrm{d}x_1\wedge\cdots\wedge \mathrm{d}x_k$$
$$= 0+\int_0^{\infty}\cdots\int_0^{\infty} \mathrm{e}^{-(x_1+\cdots+x_k)}\mathrm{d}x_1\wedge\cdots\wedge \mathrm{d}x_k$$
$$= \prod_{j=1}^{k}\int_0^{\infty}\mathrm{e}^{-x_j}\mathrm{d}x_j = \prod_{j=1}^{k}[-\mathrm{e}^{-x_j}]_0^{\infty} = 1.$$

Hence $f_5(x_1\cdots x_k)$ is a joint density function of $x_1,\cdots,x_k$.

1.4.1. The marginal and conditional densities

If $f(x_1,\cdots,x_k)$ is a joint density function of $x_1,\cdots,x_k$ then the density function of any subset of these variables, say for example, $x_1,\cdots,x_r, r\le k$, is available by integrating out the other variables. The density thus obtained is called the marginal density of that subset. For example, the marginal density of x_1 is available by integrating out $x_2,\cdots,x_k$ from $f(x_1,\cdots,x_k)$.

Example 1.4.2. Evaluate the marginal densities of the individual variables in (4) and (5) of Example 1.4.1.

Solutions 1.4.2. Integrating out x_1 in (4), observing that x_1 goes from 0 to x_2, we obtain the marginal density of x_2, denoted by $g_2(x_2)$. That is,

$$g_2(x_2)=\int_{x_1=0}^{x_2}2\mathrm{d}x=\begin{cases}2x_2, 0\le x_2\le 1\\ 0, \quad \text{elsewhere.}\end{cases} \tag{1.4.1}$$

Similarly, observing that x_2 goes from x_1 to 1, the marginal density of x_1, denoted by $g_1(x_1)$, is available as the following:

$$g_1(x_1)=\int_{x_2=x_1}^{1}2\mathrm{d}x_2=\begin{cases}2(1-x_1), 0\le x_1\le 1\\ 0, \quad \text{elsewhere.}\end{cases} \tag{1.4.2}$$

Here $g_1(x_1)$ and $g_2(x_2)$ are the marginal densities of x_1 and x_2 respectively. In (5) of Example 1.4.1 the variables are separated and the integral over x_j gives

$$\int_0^\infty \mathrm{e}^{-x_j}\mathrm{d}x_j = 1., i = 1, \cdots, k.$$

Hence the marginal density of x_j, denoted by $h_j(x_j)$, is given by,

$$h_j(x_j) = \begin{cases} \mathrm{e}^{-x_j}, 0 \le x_j < \infty \\ 0, \quad \text{elsewhere} \quad j = 1, 2, \cdots, k. \end{cases} \tag{1.4.3}$$

One interesting property may be noted from (5) of Example 1.4.1. Here the joint density is the product of the marginal densities whereas in (4) of Example 1.4.1 the joint density is not equal to the product of marginal densities.

1.4.2. Conditional densities and statistical independence.

Let $f(x_1, \cdots, x_k)$ be the joint density of the real scalar random variables $x_1, \cdots, x_k$. Consider two non-overlapping subsets of random variables, for example $\{x_1, \cdots, x_r\}$, $\{x_{r+1}, \cdots, x_k\}$, $r < k$, the two subsets need not exhaust the whole set. Let $f_1(x_1, \cdots, x_r)$ and $f_2(x_{r+1}, \cdots, x_k)$ be the marginal densities of these mutually exclusive subsets. Then the conditional density of the first subset given the second subset, denoted by $g(x_1, \cdots, x_r|x_{r+1}, \cdots, x_k)$ is defined as

$$g(x_1, \cdots, x_r|x_{r+1}, \cdots, x_k) = \frac{f(x_1, \cdots, x_k)}{f_2(x_{r+1}, \cdots, x_k)} \tag{1.4.4}$$

for $f_2(x_{r+1}, \cdots, x_k) \neq 0$ at the given points for $x_{r+1}, \cdots, x_k$.

Example 1.4.3. Evaluate the conditional density of x_1 given $x_2 = \frac{1}{3}$ in (4) of Example 1.4.1.

Solution 1.4.1. The marginal densities are available from (1.4.2) and (1.4.1) respectively. Hence the conditional density of x_1 given $x_2 = \frac{1}{3}$, denoted by $g(x_1|x_2 = \frac{1}{3})$ is given by the following:

$$\begin{aligned} g\left(x_1|x_2 = \frac{1}{3}\right) &= \frac{\text{Joint density of } x_1 \text{ and } x_2}{\text{Marginal density of } x_2}, \text{ evaluated at } x_2 = \frac{1}{3} \\ &= \frac{2}{2x_2}|_{x_2} = \frac{1}{3} = 3. \end{aligned}$$

But in the joint density x_1 goes from 0 to x_2. Hence

$$g\left(x_1|x_2=\frac{1}{3}\right)=\begin{cases}3, 0\le x_1\le \frac{1}{3}\\ 0, \quad \text{elsewhere.}\end{cases}$$

Note that here the conditional density of x_1 depends upon the condition on x_2.

Example 1.4.4. Evaluate the conditional density of x_1 given $x_2=a_2,\cdots,x_k=a_k$ in (5) of Example 1.4.1.

Solution 1.4.2. The marginal densities are available from (1.4.3). Hence the marginal joint density of $x_2,\cdots,x_k$ is $\mathrm{e}^{-(x_2,\cdots,x_k)}, 0\le x_j<\infty, j=2,\cdots,k$ and zero elsewhere. The conditional density of x_1 given $x_2,\cdots,x_k$, denoted by $g(x_1|x_2,\cdots,x_k)$ is then

$$g(x_1|x_2,\cdots,x_k)=\frac{\text{Joint density of } x_1,\ \cdots,x_k}{\text{Marginal density of } x_2,\cdots,x_k}=\frac{\mathrm{e}^{-(x_1+\cdots+x_k)}}{\mathrm{e}^{-(x_2+\cdots+x_k)}}=\mathrm{e}^{-x_1}$$
$$=\begin{cases}\mathrm{e}^{-x_1}, 0\le x_1<\infty\\ 0, \quad \text{elsewhere.}\end{cases}$$

Here, whatever be the values at which $x_2,\cdots,x_k$ are fixed or given, that has no relevance to the density of x_1. Further, note that in this case the joint density is the product of marginal densities. If such a thing happens then the variables are said to be independent or statistically independently distributed.

Definition 1.4.2. Statistical independence.

If the joint density $f(x_1,\cdots,x_k)$ is the product of the individual marginal densities of $x_1,\cdots,x_k$ then the real scalar random variables $x_1,\cdots,x_k$ are said to be *mutually independently distributed*. If this property holds in two subsets of the variables then these subsets are said to be independently distributed. In such a case the joint density of the two subsets is the product of the marginal densities of the subsets and the conditional density of one subset given the other is free of the conditions or it is the marginal density of the first subset itself. Note the following:

Joint density = (conditional density) × (marginal density of the conditioned variables)

$$f(x_1,\cdots,x_k) = g(x_1,\cdots,x_r|x_{r+1},\cdots,x_k)f_2(x_{r+1},\cdots,x_k).$$

This is a general result.

$g(x_1,\cdots,x_r|x_{r+1},\cdots,x_k) = f_1(x_1,\cdots,x_r) =$ marginal density of $\{x_1,\cdots,x_r\}$ if the two sets $\{x_1,\cdots,x_r\}$ and $\{x_{r+1},\cdots,x_k\}$ are independently distributed. When $\{x_1,\cdots,x_r\}$ and $\{x_{r+1},\cdots,x_k\}$ are independently distributed then

$$f(x_1,\cdots,x_k) = f_1(x_1,\cdots,x_r)f_2(x_{r+1},\cdots,x_k). \tag{1.4.5}$$

If $x_1,\cdots,x_k$ are mutually independently distributed then

$$f(x_1,\cdots,x_k) = \prod_{j=i}^{k} f_j(x_j) \tag{1.4.6}$$

where $f_j(x_j)$ is the marginal density x_j, $j = 1,\cdots,k$. Observe that independence in subsets need not imply mutual independence. For example x_1 and x_2 independent, x_1 and x_3 independent, x_2 and x_3 independent, need not imply that x_1, x_2, x_3 are mutually independent.

Definition 1.4.3. **Joint moments or product moments**.

$$M_{x_1,\cdots,x_k}(h_1,\cdots,h_k) = \int_{x_1}\cdots\int_{x_k} x_1^{h_1}\cdots x_k^{h_k} f(x_1,\cdots,x_k)\mathrm{d}x_1\wedge\cdots\wedge \mathrm{d}x_k \tag{1.4.7}$$

if it exists, where $f(x_1,\cdots,x_k)$ is the joint density of $x_1,\cdots,x_k$, is known as the expected value of the product $x_1^{h_1}\cdots x_k^{h_k}$, written as $E[x_1^{h_1}\cdots x_k^{h_k}]$, or the $(h_1,\cdots,h_k)^{th}$ product moment of $x_1,\cdots,x_k$. When $h_j = s_j - 1, j = 1,\cdots,k$ where $s_1,\cdots,s_k$ are arbitrary parameters, the product moment is known as the *Mellin transform* of $f(x_1,\cdots,x_k)$ when $x \geq 0, j = 1,\cdots,k$.

Definition 1.4.4. **Moment generating functions.**

Let $f(x_1,\cdots,x_k)$ be the joint density of the real scalar random variables $x_1,\cdots,x_k$ then the expected value of $\mathrm{e}^{(t_1x_1+\cdots+t_kx_k)}$, where $t_1,\cdots,t_k$, are arbitrary parameters, is known as the joint moment generating function of

$x_1,\cdots,x_k$ when the expected value exists. When t_j is replaced by $-t_j$, that is $E[\mathrm{e}^{-t_1x_1-\cdots-t_kx_k}]$ is the *Laplace transform* of $f(x_1,\cdots,x_k)$ when $x_j\ge 0, j=1,\cdots,k$. When t_j is replaced by $it_j, i=\sqrt{-1}, j=1,\cdots,k$ then the expected value is the *Fourier transform* of the density $f(x_1,\cdots,x_k)$ or the *Characteristic function* of $x_1,\cdots,x_k$.

Moment generating function: $M_f(t_1,\cdots,t_k)$

$$M_f(t_1,\cdots,t_k) = E[\mathrm{e}^{(t_1x_1+\cdots+t_kx_k)}] = \int_{x_1}\cdots\int_{x_k}\mathrm{e}^{t_1x_1+\cdots+t_kx_k}f(x_1,\cdots,x_k)\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_k. \tag{1.4.8}$$

Characteristic function = Fourier transform of f or $\phi_f(t_1,\cdots,t_k)$:

$$\phi_f(t_1,\cdots,t_k) = E[\mathrm{e}^{(it_1x_1+it_2x_2+\cdots+it_kx_k)}], i=\sqrt{-1} \tag{1.4.9}$$

$$= \int_{x_1}\cdots\int_{x_k}\mathrm{e}^{it_1x_1+\cdots+it_kx_k}f(x_1,\cdots,x_k)\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_k.$$

Laplace transform of f or $L_f(t_1,\cdots,t_k)$:

$$L_f(t_1,\cdots,t_k) = E[\mathrm{e}^{(-t_1x_1-\cdots-t_kx_k)}], \tag{1.4.10}$$

when $x_1,\cdots,x_k$ are positive variables. That is,

$$L_f(t_1,\cdots,t_k) = M_f(-t_1,\cdots,-t_k)$$

when the variables are positive or

$$L_f(t_1,\cdots,t_k) = \int_0^\infty\cdots\int_0^\infty \mathrm{e}^{(-t_1x_1-\cdots-t_kx_k)}f(x_1,\cdots,x_k)\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_k \tag{1.4.11}$$

whenever $L_f(t_1,\cdots,t_k)$ exists. Note that $\phi_f(t_1,\cdots,t_k)$ exists always.

Exercises 1.4.

1.4.1. Check whether the following are density functions. If so, evaluate c.
(1) $f_1(x)=c\mathrm{e}^{-\theta x}$; (2) $f_2(x)=c\mathrm{e}^{5x}, 0\le x<\infty$; (3) $f_3(x)=c\mathrm{e}^{-\theta|x|}, -\infty<x<\infty$;

(4) $f_4(x) = c\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x, \infty, -\infty < \mu, \infty, \sigma > 0;$

(5) $f_5(x) = \frac{\lambda_1}{\sqrt{2\pi}\sigma_1}\mathrm{e}^{-\frac{1}{2\sigma_1^2}(x-\mu_1)^2} + \frac{\lambda_2}{\sqrt{2\pi}\sigma_2}\mathrm{e}^{-\frac{1}{2\sigma_2^2}(x-\mu_2)^2}, -\infty < x < \infty,\ \lambda_1 > 0,$

$\lambda_2 > 0, \lambda_1 + \lambda_2 = 1, \sigma_1 > 0, \sigma_2 > 0, -\infty < \mu < \infty, -\infty < \mu_2 < \infty.$

1.4.2. Let $f(x)$ be a density function. Consider the distribution function or the cumulative density function of x, namely $F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t$. Let $y = F(x)$. What is the density of y?

1.4.3. Is the following a density function? If so, evaluate the marginal densities. $f(x_1, x_2) = c(x_1 + x_2), 0 \le x_1 \le 1, 0 \le x_2 \le 1$, and $f(x_1, x_2) = 0$ elsewhere.

1.4.4. Is the following a density function? If so, evaluate the probability that x falls in the interval $[1.5, 2.5]$, that is, $Pr\{1.5 \le x \le 2.5\}$ which is the integral over this interval.

$$f(x) = \begin{cases} cx, 0 \le x \le 2 & \text{and } f(x) = 0 \text{ elsewhere.} \\ (3-x), 2 \le x \le 3. \end{cases}$$

1.4.5. Check whether x_1 and x_2 in Exercise 1.4.3 are independently distributed.

1.4.6. In the joint density $f(x_1, x_2) = 2, 0 \le x_1 \le x_2 \le 1$, and $f(x_1, x_2) = 0$ elsewhere, evaluate (1) the conditional density of x_2 given $x_1 = \frac{1}{4}$; (2) the conditional density of x_1 given $x_2 = 2$.

1.4.7. Evaluate the product moment $E(x_1^2 x_2)$ in Exercise 1.4.6.

1.4.8. For the problem (1) of Exercise 1.4.1 evaluate the moment generating function. Then by using this moment generating function obtain the 4^{th} moment $E(x^4)$ by (1): expanding the moment generating function; (2): by differentiating the moment generating function.

1.4.9. For the density in Exercise 1.4.3 evaluate the joint moment generating function and show that

$$E[\mathrm{e}^{t_1x_1+t_2x_2}] = \frac{(\mathrm{e}^{t_2}-1)}{t_2}\Big[\frac{\mathrm{e}^{t_1}}{t_1} - \frac{(\mathrm{e}^{t_1}-1)}{t_1^2}\Big] + \Big(\frac{(\mathrm{e}^{t_1}-1)}{t_1}\Big)\Big[\frac{\mathrm{e}^{t_2}}{t_2} - \frac{(\mathrm{e}^{t_2}-1)}{t_2^2}\Big]; t_1 \ne 0, t_2 \ne 0.$$

1.4.10. For the normal density in (4) of Exercise 1.4.1 evaluate the *mean value* = $E(x)$ and the *variance* = $E[x-E(x)]^2$ and show that the mean value $E(x)=\mu$ and the variance = σ^2 there.

1.5. Gamma, Beta and Related Densities.

A gamma density is associated with a gamma function. A two parameter gamma density is the following:

$$f(x)=\begin{cases}\frac{x^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha)}e^{-x/\beta}; & x\ge 0,\beta>0,\alpha>0\\ 0, \text{ elsewhere.}\end{cases} \tag{1.5.1}$$

For the gamma function to exist, α can be complex also, the condition $\Re(\alpha)>0$, is required where $\Re(.)$ denotes the real part of (.).

Example 1.5.1. Evaluate the following for a gamma density: (a) The h^{th} moment of the gamma random variable x; (b) the moment generating function of x; (c) the Laplace transform of f; (d) the Fourier transform of f or the characteristic function of x.

Solution 1.5.1.

(a) The h-th moment is $E(x^h)$.

$$\begin{aligned}E(x^h)&=\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_0^{\infty}x^hx^{\alpha-1}e^{-x/\beta}dx \qquad (\text{Put } y=\frac{x}{\beta})\\ &=\beta^h\frac{\Gamma(\alpha+h)}{\Gamma(\alpha)} \quad \text{for } \Re(\alpha+h)>0.\end{aligned} \tag{1.5.2}$$

Note that h can be negative also provided $\alpha+h>0$ when α and h are real.

(b) Moment generating function of $x, M_x(t)$:

$$\begin{aligned}M_x(t)=E[e^{tx}]&=\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_0^{\infty}x^{\alpha-1}e^{tx}e^{-x/\beta}dx\\ &=\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_0^{\infty}x^{\alpha-1}e^{-(1-\beta t)x/\beta}dx<\infty \text{ for } 1-\beta t>0\\ &=(1-\beta t)^{-\alpha} \text{ for } 1-\beta t>0.\end{aligned} \tag{1.5.3}$$

Hence the Laplace transform of the gamma density is given by

$$L_f(t) = E[\mathrm{e}^{-\mathrm{t}x}] = (1 + \beta \mathrm{t})^{-\alpha}, 1 + \beta \mathrm{t} > 0. \quad (1.5.4)$$

The characteristic function of x or the Fourier transform of f:

$$\phi(t) = E[\mathrm{e}^{\mathrm{i}\mathrm{t}x}] = (1 - \mathrm{i}\beta \mathrm{t})^{-\alpha}, \Re(1 - \mathrm{i}\beta \mathrm{t}) > 0, \mathrm{i} = \sqrt{-1}. \quad (1.5.5)$$

Through the uniqueness of the Mellin and inverse Mellin transform pair, Laplace and inverse Laplace transform pair, Fourier and inverse Fourier transform pair, the density f is uniquely determined by $h = (s-1)^{th}$ moment in (1.5.2), the Laplace transform in (1.5.4) and the Fourier transform in (1.5.5).

1.5.1. The beta function and the beta density

Notation 1.5.1. $B(\alpha, \beta)$: **The beta function with parameters α and β**

Definition 1.5.1.

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \Re(\alpha) > 0, \Re(\beta) > 0. \quad (1.5.6)$$

Example 1.5.2. Derive integral representations for $B(\alpha, \beta)$ by using the integral representations of $\Gamma(\alpha)$ and $\Gamma(\beta)$.

Solution 1.5.2. From the integral representation of a gamma function in (1.1.13)

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x, \Re(\alpha) > 0$$

and

$$\Gamma(\beta) = \int_0^\infty y^{\beta-1}\mathrm{e}^{-y}\mathrm{d}y, \Re(\beta) > 0.$$

Hence,

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^\infty x^{\alpha-1}y^{\beta-1}\mathrm{e}^{-(x+y)}\mathrm{d}x \wedge \mathrm{d}y.$$

Put $x = r\cos^2\theta, y = r\sin^2\theta, 0 \le r < \infty, 0 \le \theta \le \pi/2$. Then

$$\mathrm{d}x \wedge \mathrm{d}y = 2r\cos\theta\sin\theta \;\; \mathrm{d}r \wedge \mathrm{d}\theta, x + y = r.$$

Integrating out r by using a gamma integral we obtain

$$\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha+\beta)\int_{\theta=0}^{\pi/2}(\cos^2\theta)^{\alpha-1}(\sin^2\theta)^{\beta-1}2\cos\theta\sin\theta\mathrm{d}\theta.$$

Hence

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_{\theta=0}^{\pi/2}(\cos^2\theta)^{\alpha-1}(\sin^2\theta)^{\beta-1}2\cos\theta\sin\theta\mathrm{d}\theta. \tag{1.5.7}$$

Here (1.5.7) is one integral representation. Some others are the following, the necessary transformation is written in brackets.

$$B(\alpha,\beta) = \int_{\theta=0}^{\pi/2}(\cos^2\theta)^{\alpha-1}(\sin^2\theta)^{\beta-1}2\cos\theta\sin\theta\;\mathrm{d}\theta$$

$$= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x, (x = \cos^2\theta) \tag{1.5.8}$$

$$= \int_0^1 y^{\beta-1}(1-y)^{\alpha-1}\mathrm{d}y, (y = 1-x) \tag{1.5.9}$$

$$= \int_0^\infty u^{\alpha-1}(1+u)^{-(\alpha+\beta)}\mathrm{d}u, (u = \frac{x}{1-x}) \tag{1.5.10}$$

$$= \int_0^\infty v^{\beta-1}(1+v)^{-(\alpha+\beta)}\mathrm{d}v, (v = \frac{1}{u}). \tag{1.5.11}$$

Here (1.5.8) and (1.5.9) are known as the *type-1 integral representation of the beta function* $B(\alpha,\beta)$ and (1.5.10) and (1.5.11) are known as the *type-2 representation of the beta function*. Note also that $B(\alpha,\beta) = B(\beta,\alpha)$. Based on these representations we have the type-1 and type-2 beta densities.

Definition 1.5.2. The real type-1 beta density.

$$f_1(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}, 0 \le x \le 1, \Re(\alpha) > 0, \Re(\beta) > 0 \\ 0, \text{ elsewhere.} \end{cases} \tag{1.5.12}$$

Example 1.5.3. Evaluate the h^{th} moment of the real type-1 beta random variable x and the Mellin transform of the density $f_1(x)$.

Solution 1.5.3. The h-th moment is $E(x^h)$.

$$\begin{aligned} E(x^h) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{(\alpha+h)-1}(1-x)^{\beta-1}\mathrm{d}x \\ &= \frac{\Gamma(\alpha+h)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+h)}, \Re(\alpha+h)>0, \end{aligned} \quad (1.5.13)$$

evaluated from the normalizing constant $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$ by observing that the only change is that α is replaced by $\alpha+h$. By replacing h by $s-1$ we get the Mellin transform of the density f_1. That is,

$$M_{f_1}(s) = E(x^{s-1}) = \frac{\Gamma(\alpha+s-1)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+s-1)}, \Re(\alpha+s-1)>0. \quad (1.5.14)$$

Definition 1.5.3. The real type-2 beta density

$$f_2(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1+x)^{-(\alpha+\beta)}, 0\le x<\infty, \Re(\alpha)>0, \Re(\beta)>0 \\ 0, \text{ elsewhere.} \end{cases} \quad (1.5.15)$$

This is based on the type-2 integral representation of a beta function.

Example 1.5.4. Evaluate the h-th moment of a real type-2 beta random variable with the density f_2 above.

Solution 1.5.4.

$$\begin{aligned} E(x^h) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty x^{\alpha+h-1}(1+x)^{-(\alpha+\beta)}\mathrm{d}x \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty x^{(\alpha+h)-1}(1+x)^{-[(\alpha+h)+(\beta-h)]}\mathrm{d}x \\ &= \frac{\Gamma(\alpha+h)}{\Gamma(\alpha)}\frac{\Gamma(\beta-h)}{\Gamma(\beta)} \text{ for } \Re(\alpha+h)>0, \Re(\beta-h)>0 \end{aligned} \quad (1.5.16)$$

that is, $-\Re(\alpha) < \Re(h) < \Re(\beta)$. Note that only a few moments will exist here. When α, β, h are real then h has to be between $-\alpha$ and β.

Example 1.5.5. Let x and y be independently distributed real gamma random variables having the density as in (1.5.1) with the parameters $(\alpha_1, 1)$ and $(\alpha_2, 1)$ respectively. Let $u = x + y, v = \frac{x}{x+y}$ and $w = \frac{x}{y}$. Evaluate the densities of $u, v,$ and w.

Solution 1.5.5. Since the variables x and y are independently distributed the joint density of x and y is the product of the marginal densities of x and y. The marginal densities are given as gamma densities with parameters $(\alpha_1, 1)$ and $(\alpha_2, 1)$ respectively. Hence the joint density, denoted by $f(x, y)$, is the following:

$$f(x,y) = \begin{cases} \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha_1-1} y^{\alpha_2-1}, \mathrm{e}^{-(x+y)}, 0 \le x < \infty, 0 \le y < \infty. \\ 0, \text{elsewhere.} \end{cases} \tag{1.5.17}$$

Let us make the one-to-one transformation $x = r\cos^2\theta, y = r\sin^2\theta, 0 \le r < \infty, 0 \le \theta \le \pi/2$, with the Jacobian $\mathrm{d}x \wedge \mathrm{d}y = 2r\cos\theta\sin\theta\ \mathrm{d}r \wedge \mathrm{d}\theta$, then we obtain the joint density of r and θ denoted by $g(r, \theta)$. That is,

$$g(r,\theta) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}(r\cos^2\theta)^{\alpha_1-1}(r\sin^2\theta)^{\alpha_2-1}\mathrm{e}^{-r}(2r\cos\theta\sin\theta)$$

for $0 \le r < \infty, 0 \le \theta \le \pi/2$, and $g(r, \theta) = 0$ elsewhere. Since the variables are separated, we note that r and θ are independently distributed. Further, the marginal densities of r and θ, denoted by $g_1(r)$ and $g_2(\theta)$, are given by the following:

$$g_1(r) = c_1 r^{\alpha_1+\alpha_2-1}\mathrm{e}^{-r}, 0 \le r < \infty$$

and zero elsewhere and

$$g_2(\theta) = c_2(\cos^2\theta)^{\alpha_1-1}(\sin^2\theta)^{\alpha_2-1}(2\cos\theta\sin\theta), 0 \le \theta \le \pi/2$$

and zero elsewhere, where c_1 and c_2 are the normalizing constants. Since

$$\int_0^\infty g_1(r)\mathrm{d}r = 1 \Rightarrow c_1 = \frac{1}{\Gamma(\alpha_1+\alpha_2)}$$

and

$$\int_0^{\pi/2} g_2(\theta)\mathrm{d}\theta = 1 \Rightarrow c_2 = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}.$$

Hence the nonzero part of the densities are the following:

$$g_1(r) = \frac{r^{\alpha_1+\alpha_2-1}\mathrm{e}^{-r}}{\Gamma(\alpha_1+\alpha_2)}, 0 \le r < \infty, \Re(\alpha_1+\alpha_2) > 0 \quad (1.5.18)$$

and

$$g_2(\theta) = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}(\cos^2\theta)^{\alpha_1-1}(\sin^2\theta)^{\alpha_2-1}(2\cos\theta\sin\theta), 0 \le \theta \le \pi/2. \quad (1.5.19)$$

Thus we have the following:

$$u = x + y = r\cos^2\theta + r\sin^2\theta = r$$

has a gamma density as in (1.5.18) with parameter $(\alpha_1+\alpha_2)$

$$v = \frac{x}{x+y} = \frac{r\cos^2\theta}{r\cos^2\theta + r\sin^2\theta} = \cos^2\theta.$$

Put $v = \cos^2\theta$ in (1.5.19) to obtain the density of v, denoted by $g_3(v)$, as follows:

$$g_3(v) = \begin{cases} \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} v^{\alpha_1-1}(1-v)^{\alpha_2-1}, 0 \le v \le 1, \Re(\alpha_1) > 0,\ \Re(\alpha_2) > 0 \\ 0,\ \text{elsewhere.} \end{cases} \quad (1.5.20)$$

Hence v has a real type-1 beta distribution with the parameters (α_1, α_2) as in (1.5.20)

$$w = \frac{x}{y} = \frac{r\cos^2\theta}{r\sin^2\theta} = \cot^2\theta \Rightarrow \frac{1}{1+w} = \sin^2\theta \Rightarrow$$

$$2\sin\theta\cos\theta\,\mathrm{d}\theta = \frac{1}{(1+w)^2}\,\mathrm{d}w \text{ and } \cos^2\theta = 1 - \sin^2\theta = 1 - \frac{1}{1+w} = \frac{w}{1+w}.$$

Therefore,

$$g_2(\theta)\left|\frac{\mathrm{d}\theta}{\mathrm{d}w}\right| = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\Big(\frac{w}{1+w}\Big)^{\alpha_1-1}\Big(\frac{1}{1+w}\Big)^{\alpha_2-1}\frac{1}{(1+w)^2} \Rightarrow$$

$$g_4(w) = \begin{cases} \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} w^{\alpha_1-1}(1+w)^{-(\alpha_1+\alpha_2)} & ,0 \le w < \infty \\ 0, \text{ elsewhere.} \end{cases} \tag{1.5.21}$$

Hence $w = \frac{x}{y}$ has a real type-2 beta distribution as given in (1.5.21). Thus $u = x_1 + x_2 \sim$ gamma $(\alpha_1+\alpha_2, 1), v = \frac{x}{x+y} \sim$ type-1 beta $(\alpha_1, \alpha_2), w = \frac{x}{y} \sim$ type-2 beta (α_1, α_2), where "$\sim$" indicates "distributed as " and the parameters are given in the bracket.

Exercises 1.5.

1.5.1. If $x_1, \cdots, x_k$ are mutually independently distributed real scalar random variables and if $\phi_1(x_1), \phi_2(x_2), \cdots, \phi_k(x_k)$ are functions of $x_1, \cdots, x_k$ respectively, then show that the expected value of a product is the product of the expectations that is,

$$E[\phi_1(x_1)\cdots\phi_k(x_k)] = E[\phi_1(x_1)]E[\phi_2(x_2)]\cdots E[\phi_k(x_k)].$$

1.5.2. If x is a real scalar random variable with density $f(x)$ and if a and b are constant scalars then show that (1) $E(b) = b$; (2) $E[a\phi(x)+b] = aE[\phi(x)]+b$ where $\phi(x)$ is a function of x.

1.5.3.. Show that (1) $\mathrm{Var}[ax+b] = a^2\mathrm{Var}(x)$ where a and b are constants and $\mathrm{Var}(x)$ is the variance of x.

1.5.4. Show that $\mathrm{Var}(x) = E(x^2) - [E(x)]^2$, for any real scalar random variable x.

1.5.5. The covariance between two real scalar random variables x and y is denoted and defined as $\mathrm{Cov}(x_1, x_2) = E[x - E(x)][y - E(y)]$. Show that $\mathrm{Cov}(x, y) = E(xy) - E(x)E(y)$.

1.5.6. If x has the uniform density $f(x) = \frac{1}{\theta}, 0 \le x \le \theta$ and zero elsewhere, evaluate the variance of $2x + 5$.

1.5.7. From the joint density, $f(x, y) = 2, 0 \le x \le y \le 1$ and $f(x, y) = 0$ elsewhere, evaluate $\mathrm{Cov}(x, y)$.

1.5.8. Linear correlation coefficient ρ between real scalar random variables x and y, is defined as $\rho = \frac{\text{Cov}(x,y)}{\sqrt{\text{Var}(x)\text{Var}(y)}}$ for $\text{Var}(x) \neq 0, \text{Var}(y) \neq 0$. Show that $-1 \leq \rho \leq 1$ and that $\rho = \pm 1$ if and only if x and y are linearly related.

1.5.9. Let $x_1, \cdots, x_k$ be independently distributed real type-1 beta random variables with the parameters $(\alpha_1, \beta_1) \cdots (\alpha_k, \beta_k)$. Let $u = x_1 x_2 \cdots x_k$ the product of these variables. Evaluate the h-th moment of u.

1.5.10. Let x_1 be type-1 beta with parameters $(\alpha_1, \beta_1), x_2$ be type-2 beta with parameters (α_2, β_2) and x_3 be a gamma random variable with the parameters $(\alpha_3, 1)$. Let x_1, x_2, x_3 be mutually independently distributed. Let $u = \frac{x_1 x_2}{x_3}$. Evaluate the h-th moment of u and write down the conditions for its existence.

1.6. Evaluation of Residues When Gammas are Involved

A very wide class of elementary special functions are special cases of a generalized function known as Meijer's G-function which is defined as a Mellin-Barnes integral involving certain gamma products. Explicit evaluations of such integrals and computable representations involve the evaluation of residues of functions containing gamma products. As a background training we will start with some simple exercises of evaluating residues.

Definition 1.6.1. Residue

If a is a singular point or pole of a function $f(z)$ and if $f(z)$ is expanded at the point a by using a Laurent's series, namely

$$f(z) = \sum_{-\infty}^{\infty} a_k (z-a)^k \tag{1.6.1}$$

then the coefficient a_k for $k = -1$ is called the residue of $f(z)$ at $z = a$. If a is a pole of order m then it can be shown that

$$a_{-1} = \lim_{z \to a} \left\{ \frac{1}{(m-1)!} \left[\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z-a)^m f(z) \right] \right\}. \tag{1.6.2}$$

In practice, the formula (1.6.2) is used to evaluate the residue at $z = a$ rather than looking for a Laurent's series for $f(z)$.

1.6.1. The residue theorem

In order to evaluate certain contour integrals, integrals in the complex plane, one can use a result known as the residue theorem. If C is a closed contour and $f(z)$ a function of the complex variable z which is analytic within and on C except for a countable number of singular points in the interior of C then

$$\int_C f(z)\mathrm{d}z = (2\pi i)\sum_{j=1}^{k} R_j \tag{1.6.3}$$

where $i = \sqrt{-1}$, $R_1, ..., R_k$ are the residues of $f(z)$ at the k poles within C.

Example 1.6.1. Evaluate the integral

$$g(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{x^{-z}}{(z+1)(z+2)}\mathrm{d}z,\ x > 0. \tag{1.6.4}$$

Solution 1.6.1. The contour is the infinite semicircle enclosing the left half of the complex plane. One could have taken the contour as $c - i\infty$ to $c + i\infty$ with $c > -1$. The poles of the integrand in (1.5.4) are at $z = -1$ and $z = -2$ and the residues at these points are given by

$$\begin{aligned} R_1 &= \lim_{z\to -1}(z+1)\left[\frac{x^{-z}}{(z+1)(z+2)}\right] = \lim_{z\to -1}\left[\frac{x^{-z}}{z+2}\right] \\ &= x \end{aligned}$$

and

$$\begin{aligned} R_2 &= \lim_{z\to -2}(z+2)\left[\frac{x^{-z}}{(z+1)(z+2)}\right] = \lim_{z\to -2}\left[\frac{x^{-z}}{z+1}\right] \\ &= -x^2. \end{aligned}$$

Hence

$$g(x) = x - x^2.$$

Example 1.6.2. Evaluate the integral

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)x^{-s}\mathrm{d}s,\ c > 0, x > 0.$$

Solution 1.6.2. The poles of the integrand $\Gamma(s)x^{-s}$ are coming from $\Gamma(s)$ which are at $s = 0, -1, ...$ or $s = -\nu,\ \nu = 0, 1, ...$ The residue at $s = -\nu$, denoted by R_ν, is given by

$$R_\nu = \lim_{s\to-\nu}\left[(s+\nu)\Gamma(s)x^{-s}\right].$$

In order to evaluate this limit we will use some properties of gamma functions. Write

$$\begin{aligned}(s+\nu)\Gamma(s)x^{-s} &= \frac{(s+\nu)(s+\nu-1)...s\Gamma(s)x^{-s}}{(s+\nu-1)...s}\\ &= \frac{\Gamma(s+\nu+1)x^{-s}}{(s+\nu-1)...s}.\end{aligned}$$

Then

$$\begin{aligned}\lim_{s\to-\nu}[(s+\nu)\Gamma(s)x^{-s}] &= \lim_{s\to-\nu}\frac{\Gamma(s+\nu+1)x^{-s}}{(s+\nu-1)...s}\\ &= \frac{\Gamma(1)x^\nu}{(-1)(-2)...(-\nu)} = \frac{(-1)^\nu x^\nu}{\nu!}.\end{aligned}$$

Hence

$$f(x) = \sum_{\nu=0}^{\infty} R_\nu = \sum_{\nu=0}^{\infty}\frac{(-1)^\nu x^\nu}{\nu!} = \mathrm{e}^{-x}.$$

Example 1.6.3. Evaluate the sum of the residues at the poles of the gamma product $\Gamma(z)\Gamma(z+m)$, where m is a positive integer.

Solution 1.6.3. The poles of $\Gamma(z)$ are at $z = 0, -1, -2, ..., -(m-1), -m, ...$ and those of $\Gamma(z+m)$ are at $z = -m, -(m+1), ...$ Hence the poles at $z = 0, -1, ..., -(m-1)$ are simple whereas the poles at $z = -m, -(m+1), ...$ are of order 2 each. The residue at $z = -\nu,\ \nu = 0, 1, ..., m-1$ is given by

$$R_\nu = \lim_{z\to-\nu}(z+\nu)\Gamma(z)\Gamma(z+m) = \frac{(-1)^\nu}{\nu!}\Gamma(-\nu+m), \nu = 0, 1, ..., m-1$$

as done in Example 1.6.2. For $\nu = m, m+1, \ldots$ the residue at $z = -\nu$ is given by

$$\begin{aligned}
R_\nu &= \lim_{z\to-\nu}\left\{\frac{\mathrm{d}}{\mathrm{d}z}(z+\nu)^2\Gamma(z)\Gamma(z+m)\right\},\ \nu = m, m+1, \ldots \\
&= \lim_{z\to-\nu}\left\{\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{(z+\nu)^2(z+\nu-1)^2\ldots(z+m)^2}{(z+\nu-1)^2\ldots(z+m)^2}\right.\right. \\
&\times \left.\left.\frac{(z+m-1)\ldots z\Gamma(z)\Gamma(z+m)}{(z+m-1)\ldots z}\right]\right\} \\
&= \lim_{z\to-\nu}\left\{\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\Gamma^2(z+\nu+1)}{(z+\nu-1)^2\ldots(z+m)^2(z+m-1)\ldots z}\right]\right\}.
\end{aligned}$$

In order to differentiate we use the following technique. Let

$$g(z) = \frac{\Gamma^2(z+\nu+1)}{(z+\nu-1)^2\ldots(z+m)^2(z+m-1)\ldots z}.$$

Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}g(z) &= g(z)\frac{\mathrm{d}}{\mathrm{d}z}\ln g(z) \\
&= g(z)\frac{\mathrm{d}}{\mathrm{d}z}\{2\ln\Gamma(z+\nu+1) - 2\ln(z+\nu-1) \\
&\ldots - 2\ln(z+m) - \ln(z+m-1) - \ldots - \ln z\} \\
&= g(z)\left\{2\psi(z+\nu+1) - \frac{2}{z+\nu-1}\right. \\
&\left. - \ldots - \frac{2}{z+m} - \frac{1}{z+m-1} - \ldots - \frac{1}{z}\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{z\to-\nu}\frac{\mathrm{d}}{\mathrm{d}z}\ln g(z) &= 2\psi(1) + 2\left(1 + \frac{1}{2} + \ldots + \frac{1}{\nu-m}\right) \\
&+ \left(\frac{1}{\nu-m+1} + \ldots + \frac{1}{\nu}\right) \\
&= \psi(\nu+1) + \psi(\nu-m+1)
\end{aligned}$$

by using (1.3.2). Also

$$\lim_{z\to-\nu} g(z) = \frac{(-1)^\nu(-1)^{\nu-m}\Gamma^2(1)}{\nu!(\nu-m)!} = \frac{(-1)^m}{\nu!(\nu-m)!}.$$

Therefore the sum of the residues is given by

$$\sum_{\nu=0}^{\infty} R_\nu = \sum_{\nu=0}^{m-1} \frac{(-1)^\nu}{\nu!}\Gamma(-\nu+m) \tag{1.6.5}$$

$$+ \sum_{\nu=m}^{\infty} \frac{(-1)^m}{\nu!(\nu-m)!}[\psi(\nu+1)+\psi(\nu-m+1)]. \tag{1.6.6}$$

Note that by using the conversion formula $\Gamma(-\nu+m)$, that is, a gamma with the summation symbol ν negative, can be converted to a gamma with the summation symbol ν positive, if necessary. In some simplifications such a conversion is often needed.

1.6.2. Residues when several gammas are involved

Let

$$\phi(z) = \Gamma(b_1+z)...\Gamma(b_m+z)x^{-z} \tag{1.6.7}$$

$$= h(z)x^{-z} \tag{1.6.8}$$

with

$$h(z) = \Gamma(b_1+z)...\Gamma(b_m+z). \tag{1.6.9}$$

Depending upon the values of $b_1,...,b_m$ one can expect poles of orders $1,2,...,m$ if the b_j's differ by integers. Let $z=a$ be a pole of order k for $\phi(z)$. Then the residue of $\phi(z)$ at $z=a$ is given by the following:

$$R_a = \lim_{z\to a}\left\{\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial z^{k-1}}[(z-a)^k\phi(z)]\right\} \tag{1.6.10}$$

$$= \lim_{z\to a}\left\{\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial z^{k-1}}[(z-a)^k h(z)x^{-z}]\right\}. \tag{1.6.11}$$

Note that a convenient operator can be used to take x^{-z} outside. Consider the operator

$$\left[\frac{\partial}{\partial z}+(-\ln x)\right]^{k-1} = \sum_{r=0}^{k-1}\binom{k-1}{r}(-\ln x)^{k-1-r}\frac{\partial^r}{\partial z^r}. \tag{1.6.12}$$

Then

$$\begin{aligned}
&\frac{\partial^{k-1}}{\partial z^{k-1}}[(z-a)^k h(z) x^{-z}] \\
&\qquad = x^{-z}\left[\frac{\partial}{\partial z} + (-\ln x)\right]^{k-1} [(z-a)^k h(z)] \\
&\qquad = x^{-z}\sum_{r=0}^{k-1}\binom{k-1}{r}(-\ln x)^{k-1-r}\frac{\partial^r}{\partial z^r}[(z-a)^k h(z)].
\end{aligned}$$

Let

$$B(z) = (z-a)^k h(z) \text{ and } A(z) = \frac{\partial}{\partial z}\ln B(z).$$

Then

$$\begin{aligned}
\frac{\partial^r}{\partial z^r}B(z) &= \frac{\partial^{r-1}}{\partial z^{r-1}}\left[\frac{\partial}{\partial z}B(z)\right] \\
&= \frac{\partial^{r-1}}{\partial z^{r-1}}[B(z)A(z)] \\
&= \sum_{r_1=0}^{r-1}\binom{r-1}{r_1}A^{(r-1-r_1)}(z)B^{(r_1)}(z) \qquad (1.6.13)
\end{aligned}$$

where, for example,

$$A^{(m)}(z) = \frac{\partial^m}{\partial z^m}A(z).$$

Thus

$$\begin{aligned}
R_a = \frac{x^{-a}}{(k-1)!}&\sum_{r=0}^{k-1}\binom{k-1}{r}(-\ln x)^{k-1-r}\Bigg\{\sum_{r_1=0}^{r-1}\binom{r-1}{r_1}A_0^{(r-1-r_1)} \\
&\times\sum_{r_2=0}^{r_1-1}\binom{r_1-1}{r_2}A_0^{(r_1-1-r_2)}\dots\Bigg\}B_0 \qquad (1.6.14)
\end{aligned}$$

where

$$B_0 = \lim_{z\to a}B(z) \text{ and } A_0^{(m)} = \lim_{z\to a}A^{(m)}(z). \qquad (1.6.15)$$

For convenience of computations the first few terms of the differential operator

$$\left\{\frac{\partial}{\partial z} + (-\ln x)\right\}^{\nu} B(z) = H_\nu(z)B(z) \qquad (1.6.16)$$

will be listed here explicitly, where

$$A^{(0)} = A,\ A^r = [A(z)]^r,\ A^{(m)}$$

is the m-th derivative of A, $A(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln B(z)$.

$$\begin{aligned} H_0 &= 1 \\ H_1 &= (-\ln x) + A \\ H_2 &= (-\ln x)^2 + 2(-\ln x)A + A^{(1)} + A^2 \\ H_3 &= (-\ln x)^3 + 3(-\ln x)^2 A + 3(-\ln x)(A^{(1)} + A^2) \\ &\quad + (A^{(2)} + 3A^{(1)}A + A^3). \end{aligned} \tag{1.6.17}$$

Exercises 1.6.

1.6.1. Evaluate the integral

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s) x^{-s} \mathrm{d}s$$

where

$$\phi(s) = \frac{\Gamma\left(\frac{3}{2} + s - 1\right)}{\Gamma\left(\frac{3}{2}\right)} \frac{\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{7}{2} + s - 1\right)} \frac{\Gamma(2 + s - 1)}{\Gamma(2)} \frac{\Gamma(4)}{\Gamma(4 + s - 1)}$$

for $\Re\left(\frac{1}{2} + s\right) > 0$.

1.6.2 Evaluate the integral

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{1}{3} + s\right)\Gamma\left(\frac{5}{6} + s\right)}{\Gamma\left(\frac{7}{3} + s\right)\Gamma\left(\frac{17}{6} + s\right)} x^{-s} \mathrm{d}s$$

for $c > -\frac{1}{3},\ i = \sqrt{-1},\ 0 < x < 1$.

1.6.3 Evaluate

$$f(x) = \frac{1}{3\sqrt{\pi}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(3 + s)\Gamma\left(\frac{1}{2} + s\right) x^{-s} \mathrm{d}s,$$

for $x > 0,\ c > -\frac{1}{2}$.

1.6.4 Evaluate

$$f(x) = \frac{1}{144} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(3 + s)\Gamma(4 + s) x^{-s} \mathrm{d}s,\ x > 0,\ c > -3.$$

1.6.5 Prove that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\alpha+s-1)\Gamma\left(\alpha+s-\frac{1}{2}\right)}{\Gamma(\alpha+\beta+s-1)\Gamma\left(\alpha+\beta+s-\frac{1}{2}\right)}x^{-s}\mathrm{d}s$$
$$= \frac{2^{2\beta-1}}{\Gamma(2\beta)}x^{\alpha-1}\left(1-x^{\frac{1}{2}}\right)^{2\beta-1}, \tag{1.6.18}$$
$$0<x<1,\ \Re(\alpha)>0,\ \Re(\beta)>0,\ c>-\Re(\alpha-1).$$

1.7. The Hypergeometric Function and Hypergeometric Series

A hypergeometric series with p upper parameters $a_1,...,a_p$ and q lower parameters $b_1,...,b_q$ is denoted and defined as follows:

Notation 1.7.1.

$$ {}_pF_q(a_1,...,a_p;b_1,...,b_q;z) = {}_pF_q((a)_p;(b)_q;z) = {}_pF_q \tag{1.7.1}$$
$$= \text{ a hypergeometric function.}$$

The numbers of parameters p and q are written as subscripts for F and the upper and lower parameters and the argument z are separated by semicolons. For example

${}_0F_0(\ ;\ ;z)$ means no upper or lower parameters;

${}_1F_0(\alpha;\ ;z)$ means one upper parameter α and no lower parameter;

${}_1F_1(\alpha;\beta;z)$ means one upper parameter α and one lower parameter β and the argument is z.

Definition 1.7.1.

$$ {}_pF_q(a_1,...,a_p;b_1,...,b_q;z) = \sum_{r=0}^{\infty}\frac{(a_1)_r...(a_p)_r}{(b_1)_r...(b_q)_r}\frac{z^r}{r!} \tag{1.7.2}$$

where, for example, $(a_j)_r$ stands for the Pochammer symbol

$$(a_j)_r = a_j(a_j+1)...(a_j+r-1),\ (a_j)_0 = 1,\ a_j \neq 0.$$

If an upper parameter $a_k = -m,\ m = 1,2,...,$ that is, a negative integer and if no b_j is a negative integer or zero then the series in (1.7.2) terminates into a polynomial because in this case $(a_k)_r = 0$ for $r \geq m+1$. If a $b_j,\ j = 1,...,q$ is a negative integer or zero then (1.7.2) does not make sense unless there is an a_k, $k = 1, ..., p$ such that $(a_k)_r = 0$ before $(b_j)_r = 0$. In this case also (1.7.2) terminates into a polynomial. By applying a ratio test it is obvious that the series in (1.7.2) is convergent for all z when $q \geq p$, convergent for $|z| < 1$ when $p = q + 1$, divergent when $p > q + 1$. Convergence question does not arise when (1.7.2) terminates into a polynomial. ${}_pF_q$ is taken as 1 when $z = 0$. In some cases the series is convergent for $z = 1, z = -1$. Let

$$\beta = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j. \tag{1.7.3}$$

When $p = q + 1$ the series is absolutely convergent for $|z| = 1$ if $\Re(\beta) < 0$, convergent for $z = -1$ if $0 \leq \Re(\beta) < 1$ and divergent for $|z| = 1$ if $1 \leq \Re(\beta)$.

1.7.1. Some special cases

$${}_0F_0(\ ;\ ;\pm z) = \sum_{r=0}^{\infty} \frac{(\pm z)^r}{r!} = \mathrm{e}^{\pm z}. \tag{1.7.4}$$

Thus ${}_0F_0$ is the exponential series.

$${}_1F_0(\alpha;\ ; z) = \sum_{r=0}^{\infty} (\alpha)_r \frac{z^r}{r!} = (1 - z)^{-\alpha} \text{ for } |z| < 1. \tag{1.7.5}$$

It is the binomial series. For α a negative integer the series terminates into a polynomial and in this case the condition $|z| < 1$ can be removed. ${}_1F_1(\alpha; \beta; z)$ is known as the confluent hypergeometric function and ${}_2F_1(\alpha, \beta; \gamma; z)$ for $|z| < 1$ is known as Gauss' hypergeometric function. This is the most extensively studied hypergeometric function. Consider the integral

$$I = \int_0^1 x^{a-1}(1 - x)^{c-a-1}(1 - zx)^{-b}\mathrm{d}x.$$

Expanding $(1 - zx)^{-b}$ by using a binomial series for $|z| < 1$ and integrating with the help of a type-1 beta integral one has the following:

$$
\begin{aligned}
I &= \sum_{r=0}^{\infty} \frac{(b)_r z^r}{r!} \int_0^1 x^{a+r-1}(1-x)^{c-a-1}\mathrm{d}x \\
&= \sum_{r=0}^{\infty} \frac{(b)_r z^r}{r!} \frac{\Gamma(a+r)\Gamma(c-a)}{\Gamma(c+r)},\ \Re(a)>0,\ \Re(c-a)>0 \\
&= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\ {}_2F_1(a,b;c;z),\ |z|<1.
\end{aligned}
$$

This is a very interesting integral representation, namely

$$
{}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 x^{a-1}(1-x)^{c-a-1}(1-zx)^{-b}\mathrm{d}x \tag{1.7.6}
$$

for $\Re(a) > 0$, $\Re(c-a) > 0$. Let us see what happens to the integral if $z = 1$. Then the integral can still be evaluated by using a type-1 beta integral provided $\Re(c-a-b) > 0$. That is,

$$
\begin{aligned}
{}_2F_1(a,b;c;1) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \frac{\Gamma(a)\Gamma(c-a-b)}{\Gamma(c-b)} \\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}
\end{aligned} \tag{1.7.7}
$$

for $\Re(c-a-b) > 0$, $\Re(c-a) > 0$, $\Re(c-b) > 0$, $\Re(c) > 0$. This is known as the summation formula for a ${}_2F_1$ at $z = 1$. Some such summation formulae are available for ${}_2F_1$ at $z = -1$, ${}_3F_2$ at $z = 1$, ${}_4F_3$ at $z = 1$ and so on. In these formulae there are some restrictions on the parameters. In the summation formula (1.7.7) the three parameters a, b, c are free parameters, not connected to each other.

1.7.2. Mellin-Barnes representation

We have already discussed a Mellin-Barnes representation for an exponential function of the following form:

$$
\mathrm{e}^{-z} = {}_0F_0(\ ;\ ;-z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) z^{-s}\mathrm{d}s,\ |z| < \infty. \tag{1.7.8}
$$

Let us see whether we can find similar representations for ${}_1F_0$, ${}_1F_1$, ${}_0F_1$, ${}_2F_1$, ${}_2F_2$ and so on. By evaluating the sum of residues at the poles of $\Gamma(s)$ we have the following formula:

$$\begin{aligned} {}_1F_0(a;\ ;z) &= (1-z)^{-a},\ |z|<1 \\ &= \frac{1}{\Gamma(a)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(a-s)(-z)^{-s}\mathrm{d}s \end{aligned} \tag{1.7.9}$$

with $|\arg(-z)| < \pi$. Again by evaluating the sum of the residues at the poles of $\Gamma(s)$ one has the hypergeometric series of (1.7.2) from the following Mellin-Barnes representation:

$$\begin{aligned} {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) &= \left\{\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}\right\} \\ \times \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s) &\left\{\frac{\prod_{j=1}^{p}\Gamma(a_j-s)}{\prod_{j=1}^{q}\Gamma(b_j-s)}\right\}(-z)^{-s}\mathrm{d}s. \end{aligned} \tag{1.7.10}$$

Exercises 1.7.

1.7.1. Incomplete gamma function $\gamma(a,x)$, $\Gamma(a,x)$. Show that

$$(a)\ \gamma(a,x) = \int_0^x t^{a-1}\mathrm{e}^{-t}\mathrm{d}t = a^{-1}x^a\ {}_1F_1(a;a+1;-x),\ \Re(a)>0; \tag{1.7.11}$$

$$(b)\ \Gamma(a,x) = \int_x^\infty t^{a-1}\mathrm{e}^{-t}\mathrm{d}t = \Gamma(a)-\gamma(a,x). \tag{1.7.12}$$

1.7.2. Incomplete beta functions $B_x(\alpha,\beta), I_x(\alpha,\beta)$. Show that

$$\begin{aligned} (a)\ B_x(\alpha,\beta) &= \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t,\ \Re(\alpha)>0,\ \Re(\beta)>0 \\ &= \alpha^{-1}x^\alpha\ {}_2F_1(\alpha,1-\beta;\alpha+1;x),\ 0<x<1; \end{aligned} \tag{1.7.13}$$

$$(b)\ I_x(\alpha,\beta) = \frac{B_x(\alpha,\beta)}{B_1(\alpha,\beta)},\ B_1(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.7.14}$$

1.7.3. Whittaker function $M_{\mu,\nu}(z)$. Show that

$$M_{\mu,\nu}(z) = z^{\nu+\frac{1}{2}} e^{-z/2}\ {}_1F_1\left(\frac{1}{2} - \mu + \nu; 2\nu + 1; z\right);$$
$$= \frac{\Gamma(1+2\nu)}{\Gamma\left(\frac{1}{2}+\nu-\mu\right)}\ e^{-z/2} z^{\nu+\frac{1}{2}}$$
$$\times \frac{1}{2\pi_i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma\left(\frac{1}{2}+\nu-\mu-s\right)}{\Gamma(1+2\nu-s)} (-z)^{-s} ds \qquad (1.7.15)$$

for $|\arg z| < \pi/2,\ 2\nu \neq -1, -2, \ldots$.
[Take the second statement as the definition and establish the first statement].

1.7.4. Whittaker function $W_{\mu,\nu}(z)$**.** Show that

$$W_{\mu,\nu}(z) = \frac{\Gamma(-2\nu)}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)} M_{\mu,\nu}(z) + \frac{\Gamma(2\nu)}{\Gamma\left(\frac{1}{2}-\mu+\nu\right)} M_{\mu,-\nu}(z), \qquad (1.7.16)$$
$$\frac{1}{2} - \mu \pm \nu \neq 0, -1, -2, \ldots, 2\nu \neq 0, \pm 1, \pm 2, \ldots;$$
$$= \frac{z^{\mu} e^{-z/2}}{\Gamma\left(\frac{1}{2}+\nu-\mu\right)} \int_0^{\infty} e^{-t} t^{\nu-\mu-\frac{1}{2}} \left(1 + \frac{t}{z}\right)^{\nu+\mu-\frac{1}{2}} dt \qquad (1.7.17)$$
$$\Re\left(\frac{1}{2}+\nu-\mu\right) > 0 \text{ and } |\arg z| < \pi;$$
$$= \frac{z^{\mu} e^{-z/2}}{\Gamma\left(\frac{1}{2}+\nu-\mu\right)\Gamma\left(\frac{1}{2}-\nu-\mu\right)}$$
$$\times \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(-s)\Gamma\left(\frac{1}{2}+\nu-\mu+s\right)\Gamma\left(\frac{1}{2}-\nu-\mu+s\right) z^{-s} ds, \qquad (1.7.18)$$
$$|\arg z| < \frac{3\pi}{2},\ -\frac{1}{2} + \mu \pm \nu \neq 0, 1, 2, \ldots$$

[Take the last line as the definition and establish the other two].

1.7.5. Bessel functions $J_\nu(z),\ I_\nu(z),\ Y_\nu(z),\ K_\nu(z)$:

(a) Taking

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\ {}_0F_1\left(; 1+\nu; -\frac{z^2}{4}\right) \qquad (1.7.19)$$

show that

(1.7.20)

$$J_\nu(z) = \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{\nu+s}{2}\right)}{\Gamma\left(1+\frac{\nu-s}{2}\right)}\left(\frac{z}{2}\right)^{-s} \mathrm{d}s, \tag{1.7.21}$$

for $-\Re(\nu) < 1,\ |\arg z| < \pi$.

(b) Taking

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left(;1+\nu;\frac{z^2}{4}\right) \tag{1.7.22}$$

show that

$$I_\nu(z) = \mathrm{e}^{-i\nu\pi/2} J_\nu\left(z\mathrm{e}^{i\pi/2}\right),\ -\pi < \arg z \le \pi/2. \tag{1.7.23}$$

(c) Taking

$$K_\nu(z) = \left(\frac{2z}{\pi}\right)^{-\frac{1}{2}} W_{0,\nu}(2z) \tag{1.7.24}$$

show that

$$K_\nu(z) = \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma\left(s+\frac{\nu}{2}\right)\Gamma\left(s-\frac{\nu}{2}\right)\left(\frac{z^2}{4}\right)^{-s} \mathrm{d}s,\ |\arg z| < \pi/2. \tag{1.7.25}$$

For other elementary functions such as parabolic cylinder functions, Bateman's functions, Coulomb wave functions, error functions, Kelvin's functions, Lommel's functions, elliptic functions, Struve's functions, Anger-Weber functions, Neumann polynomials, theta functions and orthogonal polynomials such as Jacobi polynomials, Legendre polynomials, Gegenbauer polynomials, Chebyshev's polynomials, Laguerre polynomials, Hermite polynomials etc see books on special functions, for example, Carlson, B.C. (1977) *Special Functions of Applied Mathematics*, Academic Press, New York; Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G. (1953) *Higher Transcendental Functions, Vol.II*, McGraw-Hill, New York; Mathai, A.M. (1993) *A Handbook of Generalized Special Functions for Statistical and Physical Sciences*, Oxford University Press, Oxford; Slater, L.J. (1966) *Generalized Hypergeometric Functions*, Cambridge University Press, London.

Exercises 1.7.

1.7.6. For a Gauss' hypergeometric function ${}_2F_1$ derive the following relationships.

$$ {}_2F_1(a,b;c;z) = (1-z)^{-b}{}_2F_1(c-a,b;c;-\frac{z}{1-z}), z \neq 1 \tag{1.7.26}$$
$$ = (1-z)^{-a}{}_2F_1(a,c-b;c;-\frac{z}{1-z}), z \neq 1 \tag{1.7.27}$$
$$ = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z). \tag{1.7.28}$$

1.7.7. Let x_1 and x_2 be independently distributed real scalar gamma random variables with the parameters $(\alpha_1, 1)$ and $(\alpha_2, 1)$ respectively. Let $u = x_1x_2$. Evaluate the density of u by using Mellin transformation technique when α_1 and α_2 do not differ by integers or zero.

1.7.8. Let x_1 and x_2 be independently distributed real type-1 beta random variables with the parameters (α_1, β_1) and (α_2, β_2) respectively. Let $u = x_1x_2$. Evaluate the density of u by using Mellin transform technique if α_1 and α_2 do not differ by integers or zero.

1.7.9. Repeat the problem in Exercise 1.7.7 if x_1 and x_2 are type-2 beta distributed, where $\alpha_1 - \alpha_2 \neq \pm\lambda, \lambda = 0, 1, \cdots \beta_1 - \beta_2 \neq \pm\nu, \nu = 0, 1, 2, \cdots$.

1.7.10. Let $f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(\alpha - s)\Gamma(s)x^{-s}\mathrm{d}s$. Evaluate $f(x)$ as the sum of residues at the poles of $\Gamma(s)$. Then evaluate it again at the poles of $\Gamma(\alpha - s)$. Then compare the two results. In the first case we get the function for $|x| < 1$ and in the case for $|x| > 1$.

1.7.11. Bose-Einstein density in physics: Let

$$f(x) = \frac{1}{c[-1 + \mathrm{e}^{\alpha+\beta x}]}, 0 \leq x < \infty, \beta > 0.$$

Evaluate the normalizing constant c in this Bose-Einstein density.

1.7.12. Fermi-Dirac density in physics: Let

$$f(x) = \frac{1}{c[1 + \mathrm{e}^{\alpha+\beta x}]}, \beta > 0, 0 \leq x < \infty.$$

Evaluate the normalizing constant c.

1.7.13. Maxwell-Boltzmann density in physics: Let

$$f(x) = cx^2\mathrm{e}^{-\beta x^2}, 0 \leq x < \infty, \beta > 0.$$

Evaluate c.

1.7.14. Generalized real gamma density. Let

$$f(x) = cx^{\alpha-1}\mathrm{e}^{-bx^{\delta}}, b > 0, 0 \leq x < \infty.$$

Evaluate the normalizing constant c.

1.8. The G-function or Meijer's G-function

It is a very general function where most of the elementary special functions are its particular cases. It has four sets of parameters. Hence we will use the following convenient notation:

Notation 1.8.1.

$$G^{m,n}_{p,q}\left(z\Big|^{a_1,\dots,a_p}_{b_1,\dots,b_q}\right) = G^{m,n}_{p,q}\left(z\Big|^{(a_p)}_{(b_q)}\right) = G^{m,n}_{p,q}(z).$$

Definition 1.8.1. In terms of a Mellin-Barnes integral it is defined as follows:

$$G^{m,n}_{p,q}\left[z\Big|^{a_1,\dots,a_p}_{b_1,\dots,b_q}\right] = \frac{1}{2\pi i}\int_L \phi(s)z^{-s}\mathrm{d}s \tag{1.8.1}$$

where

$$\phi(s) = \frac{\left\{\prod_{j=1}^{m}\Gamma(b_j+s)\right\}\left\{\prod_{j=1}^{n}\Gamma(1-a_j-s)\right\}}{\left\{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)\right\}\left\{\prod_{j=n+1}^{p}\Gamma(a_j+s)\right\}}. \tag{1.8.2}$$

The contour L separates the poles of $\Gamma(b_j+s),\ j=1,\dots,m$ to those of $\Gamma(1-a_j-s),\ j=1,\dots\dots,n.$ There are three paths L available.

Path 1: L goes from $c-i\infty$ to $c+i\infty$ so that all the poles of $\Gamma(1-a_k-s),\ k=1,\dots,n$ lie to the right of the path and all poles of $\Gamma(b_j+s),\ j=1,\dots,m$ lie to the left of the path. Let

$$\delta = m+n-\frac{1}{2}(p+q),\ \mu = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j. \tag{1.8.3}$$

The integral converges for $|\arg z| < \delta\pi,\ \delta > 0$. If $|\arg z| = \delta\pi,\ \delta \geq 0$ then the integral converges absolutely for $p = q$ and $\Re(\mu) < -1$. If $|\arg z| = \delta\pi,\ \delta \geq 0,\ p \neq q$ then the integral converges absolutely for $(p-q)\sigma > \Re(\mu) + 1 - \frac{1}{2}(q-p)$ where $c_1 < \sigma < c_2$ with c_1 and c_2 as shown in Figure 1.8.1.

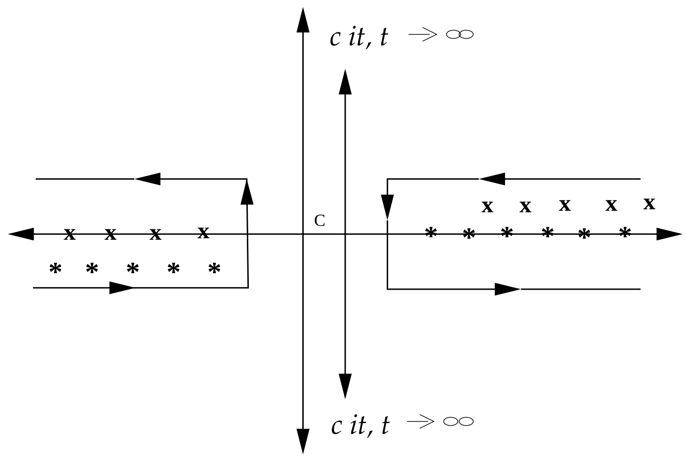

Figure 1.8.1

Path 2: L is a loop beginning and ending at $-\infty$ and encircling all the poles of $\Gamma(b_j + s),\ j = 1, ..., m$ once in the positive direction but none of the poles of $\Gamma(1 - a_k - s),\ k = 1, ..., n$. The integral converges for all z if $q \geq 1$ and $q > p$ or for $|z| < 1$ when $p = q$.

Path 3: L is a loop beginning and ending at $+\infty$ and encircling all the poles of $\Gamma(1 - a_j - s),\ j = 1, 2, ..., n$ once in the negative direction but none of the poles of $\Gamma(b_k + s),\ k = 1, ..., m$. The integral converges for all z if $p \geq 1$ and $p > q$ or for $|z| > 1$ when $p = q$.

Verification of the conditions require expansion of the gammas in (1.8.2) with the help of the asymptotic formula for gamma functions and then looking at the asymptotic equivalence of the integrand. The details may be seen from Mathai (1993) (Oxford University Press book).

The simplified conditions are the following: $G(z)$ exists for the following situations:

$$\begin{array}{rl} \text{(i)} & q \geq 1, q > p, \text{ for all } z, z \neq 0 \\ \text{(ii)} & q \geq 1, q = p, \text{ for } |z| < 1 \\ \text{(iii)} & p \geq 1, p > q, \text{ for all } z, z \neq 0 \\ \text{(iv)} & p \geq 1, p = q, \text{ for } |z| > 1. \end{array} \tag{1.8.4}$$

Example 1.8.1. Evaluate

$$f(x) = G_{1,1}^{1,0}\left[x\Big|_{\alpha}^{\alpha+\beta+1}\right].$$

Solution 1.8.1. As per our notation, $m = 1, n = 0, p = 1, q = 1$.

$$G_{1,1}^{1,0}\left[x\Big|_{\alpha}^{\alpha+\beta+1}\right] = \frac{1}{2\pi i}\int_L \frac{\Gamma(\alpha+s)}{\Gamma(\alpha+\beta+1+s)}x^{-s}\mathrm{d}s.$$

As per situation (ii) above we should obtain a convergent function for $|x| < 1$ if we evaluate the integral as the sum of the residues at the poles of $\Gamma(\alpha+s)$. The poles are at $s = -\alpha - \nu, \nu = 0, 1, \cdots$ and the sum of the residues

$$\sum_{\nu=0}^{\infty}\mathfrak{R}_{\nu} = \sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\frac{x^{\nu+\alpha}}{\Gamma(\beta+1-\nu)}; \Gamma(\beta+1-\nu) = \frac{(-1)^{\nu}\Gamma(\beta+1)}{(-\beta)_{\nu}}.$$

$$G_{1,1}^{1,0}[x\Big|_{\alpha}^{\alpha+\beta+1}] = \frac{x^{\alpha}}{\Gamma(\beta+1)}\sum_{\nu=0}^{\infty}\frac{(-\beta)_{\nu}x^{\nu}}{\nu!} = \frac{x^{\alpha}}{\Gamma(\beta+1)}(1-x)^{\beta}, |x| < 1 \tag{1.8.5}$$

for $\mathfrak{R}(\beta+1) > 0$.

Example 1.8.2. Let $u = x_1x_2\cdots x_p$ where $x_1, \cdots, x_p$ are independently distributed real random variables with (1) : x_j gamma distributed with parameters $(\alpha_j, 1), j = 1, \cdots, p$; (2) : x_j type-1 beta distributed with parameters $(\alpha_j, \beta_j), j = 1, \cdots, p$; (3) : x_j is type-2 beta distributed with parameters $(\alpha_j, \beta_j), j = 1, \cdots, p$. Evaluate the density of u in (1),(2) and (3).

Solution 1.8.2. Taking the $(s-1)^{th}$ moment of u or the Mellin transform of the density of u we have the following:

$$E(u^{s-1}) = E(x_1 \cdots x_p)^{s-1} = E(x_1^{s-1} \cdots x_p^{s-1}) = E(x_1^{s-1}) \cdots E(x_p^{s-1})$$

due to independence

$$= \prod_{j=1}^{p} E(x_j^{s-1}) = \prod_{j=1}^{p} \frac{\Gamma(\alpha_j + s - 1)}{\Gamma(\alpha_j)}, \Re(\alpha_j + s - 1) > 0, j = 1, \cdots, p$$

for case (1)

$$= \prod_{j=1}^{p} \frac{\Gamma(\alpha_j + s - 1)}{\Gamma(\alpha_j)} \frac{\Gamma(\alpha_j + \beta_j)}{\Gamma(\alpha_j + \beta_j + s - 1)}, \Re(\alpha_j + s - 1) > 0, j = 1, \cdots, p$$

for case (2)

$$= \prod_{j=1}^{p} \frac{\Gamma(\alpha_j + s - 1)}{\Gamma(\alpha_j)} \frac{\Gamma(\beta_j - s + 1)}{\Gamma(\beta_j)}, \Re(\alpha_j + s - 1) > 0, \Re(\beta_j - s + 1) > 0,$$

$j = 1, \cdots, p$ for case (3).

Let the densities be denoted by $g_1(u), g_2(u)$ and $g_3(u)$ respectively. They are available from the respective inverse Mellin transforms which can be written as G-functions as follows:

$$\begin{aligned} g_1(u) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left\{ \prod_{j=1}^{p} \frac{\Gamma(\alpha_j - 1 + s)}{\Gamma(\alpha_j)} \right\} u^{-s} \mathrm{d}s \\ &= \frac{1}{\left\{ \prod_{j=1}^{p} \Gamma(\alpha_j) \right\}} G_{0,p}^{p,0}[u|_{\alpha_j - 1, j=1,\cdots,p}], \text{ for } u > 0, \Re(\alpha_j) > 0, j = 1, \cdots, p \end{aligned} \tag{1.8.6}$$

and zero elsewhere.

$$g_2(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{\prod_{j=1}^{p}\frac{\Gamma(\alpha_j+s-1)}{\Gamma(\alpha_j)}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j+\beta_j+s-1)}\right\}u^{-s}\mathrm{d}s$$

$$= \left\{\prod_{j=1}^{p}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)}\right\}G_{p,p}^{p,0}[u\big|_{\alpha_j-1,j=1,\cdots,p}^{\alpha_j+\beta_j-1,j=1,\cdots,p}], 0 < u < 1, \tag{1.8.7}$$

$\Re(\alpha_j) > 0, \Re(\beta_j) > 0,$ and zero elsewhere .

$$g_3(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{\prod_{j=1}^{p}\frac{\Gamma(\alpha_j+s-1)}{\Gamma(\alpha_j)}\frac{\Gamma(\beta_j-s+1)}{\Gamma(\beta_j)}\right\}u^{-s}\mathrm{d}s \tag{1.8.8}$$

$$= \frac{1}{\left\{\prod_{j=1}^{p}\Gamma(\alpha_j)\Gamma(\beta_j)\right\}}G_{p,p}^{p,p}[u\big|_{\alpha_j-1,j=1,\cdots,p}^{-\beta_j,j=1,\cdots,p}], u > 0,$$

$\Re(\alpha_j) > 0, \Re(\beta_j) > 0, j = 1,\cdots,p$ and zero elsewhere.

Example 1.8.3. Evaluate the following integral, a particular case of which is the reaction rate integral in astrophysics.

$$I(\alpha,a,b,\rho) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-ax-bx^{-\rho}}\mathrm{d}x, a > 0, b > 0, \rho > 0. \tag{1.8.9}$$

Solution 1.8.3. Since the integrand can be taken as a product of positive integrable functions we can apply statistical distribution theory to evaluate this integral or such similar integrals. The procedure to be discussed here is suitable for a wide variety of problems. Let x_1 and x_2 be two real scalar random variables with density functions $f_1(x_1)$ and $f_2(x_2)$. Let $u = x_1x_2$ and let x_1 and x_2 be independently distributed. Then the joint density of x_1 and x_2, denoted by $f(x_1, x_2)$, is the product of the marginal densities due to statistical independence of x_1 and x_2. That is,

$$f(x_1, x_2) = f_1(x_1)f_2(x_2).$$

Consider the transformation $u = x_1x_2$ and $v = x_1 \Rightarrow \mathrm{d}x_1 \wedge \mathrm{d}x_2 = \frac{1}{v}\mathrm{d}u \wedge \mathrm{d}v$. Hence the joint density of u and v, denoted by $g(u, v)$, is available as,

$$g(u,v) = \frac{1}{v} f_1(v) f_2(\frac{u}{v}). \tag{1.8.10}$$

Then the density of u denoted by $g_1(u)$, is available by integrating out v from $g(u,v)$. That is,

$$g_1(u) = \int_v \frac{1}{v} f_1(v) f_2(\frac{u}{v}) \mathrm{d}v. \tag{1.8.11}$$

Here (1.8.10) and (1.8.11) are general results and the method described here is called the *method of transformation of variables* for obtaining the density of $u = x_1 x_2$. Now, consider (1.8.9). Let

$$f_1(x_1) = c_1 x_1^{\alpha} \mathrm{e}^{-a x_1} \text{ and } f_2(x_2) = c_2 \mathrm{e}^{-z x_2^{\rho}}, 0 \le x_1 < \infty, 0 \le x_2 < \infty \tag{1.8.12}$$

$a > 0, z > 0$, where c_1 and c_2 are the normalizing constants. These normalizing constants can be evaluated by using the property.

$$1 = \int_0^{\infty} f_1(x_1) \mathrm{d}x_1 \text{ and } 1 = \int_0^{\infty} f_2(x_2) \mathrm{d}x_2.$$

Since we do not need the explicit forms of c_1 and c_2 we will not evaluate them here. With the f_1 and f_2 in (1.8.12) let us evaluate (1.8.11). We have

$$g_1(u) = c_1 c_2 \int_{v=0}^{\infty} \frac{1}{v} v^{\alpha} \mathrm{e}^{-av} \mathrm{e}^{-z(\frac{u}{v})^{\rho}} \mathrm{d}v = c_1 c_2 \int_{v=0}^{\infty} v^{\alpha-1} \mathrm{e}^{-av} \mathrm{e}^{-(zu^{\rho}) v^{-\rho}} \mathrm{d}v. \tag{1.8.13}$$

Note that with $b = zu^{\rho}$, (1.8.13) is (1.8.9) multiplied by c_1 and c_2. Thus, we have identified the integral to be evaluated as a constant multiple of the density of u. This density of u is unique. Let us evaluate the density through Mellin and inverse Mellin transform technique.

$$E(u^{s-1}) = E(x_1^{s-1}) E(x_2^{s-1})$$

due to statistical independence of x_1 and x_2. But

$$E(x_1^{s-1}) = c_1 \int_0^{\infty} x_1^{\alpha+s-1} \mathrm{e}^{-a x_1} \mathrm{d}x_1 = c_1 a^{-(\alpha+s)} \Gamma(\alpha+s), \Re(\alpha+s) > 0 \tag{1.8.14}$$

and

$$E(x_2^{s-1}) = c_2 \int_0^{\infty} x_2^{s-1} \mathrm{e}^{-z x_2^{\rho}} \mathrm{d}x_2 = \frac{c_2}{\rho z^{s/\rho}} \int_0^{\infty} y^{\frac{s}{\rho}-1} \mathrm{e}^{-y} \mathrm{d}y = \frac{c_2}{\rho z^{s/\rho}} \Gamma(\frac{s}{\rho}), \Re(s) > 0. \tag{1.8.15}$$

Hence

$$E(u^{s-1}) = c_1 c_2 \frac{a^{-\alpha}}{\rho}(az^{\frac{1}{\rho}})^{-s}\Gamma(\alpha+s)\Gamma(\frac{s}{\rho}), \tag{1.8.16}$$

Therefore, the density of u, denoted by $g_1(u)$, is available from the inverse Mellin transform.

$$g_1(u) = c_1 c_2 \frac{a^{-\alpha}}{\rho}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\alpha+s)\Gamma(\frac{s}{\rho})(az^{\frac{1}{\rho}}u)^{-s}\,\mathrm{d}s. \tag{1.8.17}$$

Now, compare (1.8.17) with (1.8.13) to obtain the following:

$$\int_0^\infty \nu^{\alpha-1}\mathrm{e}^{-a\nu}\mathrm{e}^{-(zu^\rho)\nu^{-\rho}}\mathrm{d}\nu = \frac{a^{-\alpha}}{\rho}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\alpha+s)\Gamma(\frac{s}{\rho})(az^{\frac{1}{\rho}}u)^{-s}\,\mathrm{d}s. \tag{1.8.18}$$

On the right side in (1.8.18) the coefficient of s in $\Gamma(\frac{s}{\rho})$ is $\frac{1}{\rho} \neq 1$. Hence (1.8.18) is not a G-function but it can be written as an H-function, which will be considered next. In reaction rate theory in physics $\rho = \frac{1}{2}$ and then

$$\Gamma(\frac{s}{\rho}) = \Gamma(2s) = \pi^{\frac{1}{2}}2^{2s-1}\Gamma(s)\Gamma\Big(s+\frac{1}{2}\Big)$$

by using the duplication formula for gamma functions. Then the right side of (1.8.18) reduces to

$$\frac{1}{2\rho a^\alpha \pi^{\frac{1}{2}}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\alpha+s)\Gamma(s)\Gamma\Big(s+\frac{1}{2}\Big)\Big(\frac{auz^{1/\rho}}{4}\Big)^{-\rho}\mathrm{d}s$$
$$= \frac{1}{2\rho a^\alpha \pi^{\frac{1}{2}}}G_{0,3}^{3,0}\Big[\frac{auz^{1/\rho}}{4}\Big|_{\alpha,0,\frac{1}{2}}\Big], u > 0.$$

But

$$b = zu^\rho \Rightarrow \frac{auz^{1/\rho}}{4} = \frac{ab^{1/\rho}}{4}.$$

Hence, for $\rho = \frac{1}{2}$,

$$\int_0^\infty \nu^{\alpha-1}\mathrm{e}^{-a\nu-b\nu^{-\rho}}\mathrm{d}\nu = \frac{1}{2\rho a^\alpha \pi^{\frac{1}{2}}}G_{0,3}^{3,0}\Big[\frac{ab^{1/\rho}}{4}\Big|_{\alpha,0,\frac{1}{2}}\Big] \text{ for } \rho = \frac{1}{2}$$
$$= \frac{1}{a^\alpha \pi^{\frac{1}{2}}}G_{0,3}^{3,0}\Big[\frac{ab^2}{4}\Big|_{\alpha,0,\frac{1}{2}}\Big], u > 0. \tag{1.8.19}$$

Exercises 1.8.

Write down the Mellin-Barnes representations in Exercises 1.8.1.- 1.8.5 where the series forms are given. Here is an illustration.

$$\begin{aligned} {}_1F_0(\alpha;;x) &= \sum_{r=0}^{\infty}(\alpha)_r\frac{x^r}{r!} = \frac{1}{\Gamma(\alpha)}\sum_{r=0}^{\infty}\Gamma(\alpha+r)\frac{x^r}{r!} \\ &= \frac{1}{\Gamma(\alpha)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\alpha-s)\Gamma(s)(-x^{-s})\mathrm{d}s. \end{aligned}$$

The last expression is the Mellin-Barnes representation for the series form ${}_1F_0(\alpha;;x)$.

1.8.1. ${}_0F_0(;;-z) = \mathrm{e}^{-z} = \sum_{r=0}^{\infty}\frac{(-z)^r}{r!}$ (exponential Series)

1.8.2. ${}_2F_1(a,b;c;z) = \sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r}\frac{(z)^r}{r!}$ (Gauss′ hypergeometric series)

1.8.3. ${}_1F_1(a;b;z) = \sum_{r=0}^{\infty}\frac{(a)_r}{(b)_r}\frac{(z)^r}{r!}$ (confluent hypergeometric series)

1.8.4. $\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{(z/2)^{\nu+2r}}{\Gamma(\nu+r+1)}$ (Bessel function $J_\nu(z)$)

1.8.5. $\sum_{r=0}^{\infty}\frac{(z/2)^{\nu+2r}}{r!\Gamma(\nu+r+1)}$ (Bessel function $I_\nu(z)$).

Write the series forms from the Mellin-Barnes representation in Exercise 1.8.6 and list the conditions for convergence and existence also.

1.8.6. $\frac{\Gamma(1+2\nu)}{\Gamma(\frac{1}{2}+\nu-\mu)}\mathrm{e}^{-z/2}z^{\nu+\frac{1}{2}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\Gamma(\frac{1}{2}+\nu-\mu-s)}{\Gamma(1+2\nu-s)}(-z)^{-s}\mathrm{d}s$ (Whittaker function $M_{\mu,\nu}(z)$)

Represent the following in Exercises 1.8.7 to 1.8.10 as G-functions and write down the conditions.

1.8.7. $z^\beta(1+az^\alpha)^{-1}$

1.8.8. $z^\beta(1+az^\alpha)^{-\gamma}$

1.8.9. (a) $\sin z$; (b) $\cos z$; (c) $\sinh z$; (d) $\cosh z$

1.8.10. (a) $\ln(1\pm z)$; $(b)\ln\left(\frac{1+z}{1-z}\right)$.

1.9. The H-function

This function is a generalization of the G-function. This was available in the literature as a Mellin-Barnes integral but Charles Fox made a detailed study of it in 1960's and hence the function is called Fox's H-function. The Mellin-Barnes representation is the following:

Notation 1.9.1. **H-function**

$$H^{m,n}_{p,q}[z|^{(a_1,\alpha_1),\cdots,(a_p,\alpha_p)}_{(b_1,\beta_1),\cdots,(b_q,\beta_q)}] = H^{m,n}_{p,q}[z|^{[(a_p,\alpha_p)]}_{[(b_q,\beta_q)]_q}] = H^{m,n}_{p,q}(z) = H(z).$$

Definition 1.9.1.

$$H^{m,n}_{p,q}[z|^{(a_1,\alpha_1),\cdots,(a_p,\alpha_p)}_{(b_1,\beta_1),\cdots,(b_q,\beta_q)}] = \frac{1}{2\pi i}\int_L \frac{\left\{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\right\}\left\{\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)\right\}}{\left\{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\right\}\left\{\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)\right\}} z^{-s}\mathrm{d}s \tag{1.9.1}$$

where $\alpha_1,\cdots,\alpha_p,\beta_1,\cdots,\beta_q$ are real positive numbers (integers, rationals or irrationals), a_j's and b_j's are, in general, complex quantities, $i=\sqrt{-1}$ and the contour L separates the poles of $\Gamma(b_j+\beta_j s)$, $j=1,\cdots,m$ from those of $\Gamma(1-a_j-\alpha_j s)$, $j=1,\cdots,n$. Three paths L, similar to the ones for a G-function, can be given for the H-function also. Details of the existence conditions, various properties and applications may be seen from Mathai and Saxena (1978) and Mathai (1993). A simplified set of existence conditions is the following: Let,

$$\mu = \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j \text{ and } \beta = \left\{\prod_{j=1}^{q}\alpha_j^{\alpha_j}\right\}\left\{\prod_{j=1}^{q}\beta_j^{-\beta_j}\right\}. \tag{1.9.2}$$

The H-function exists for the following cases:

$$\begin{aligned} &\text{(i)} \quad q\ge 1, \mu>0, \text{ for all } z, z\ne 0\\ &\text{(ii)} \quad q\ge 1, \mu=0, \text{ for } |z|<\beta^{-1}\\ &\text{(iii)} \quad p\ge 1, \mu<0, \text{ for all } z, z\ne 0\\ &\text{(iv)} \quad p\ge 1, \mu=0, \text{ for } |z|, z>\beta^{-1}. \end{aligned} \tag{1.9.3}$$

Two special cases, which follow from the definition itself, may be noted. When $\alpha_1 = 1 = \cdots = \alpha_p = \beta_1 = 1 = \cdots = \beta_q$ then the H-function reduces to a G-function. When all the α_j's and β_j's are rational numbers, that is ratios of two positive integers since by definition the α_j's and β_j's are positive real numbers, we may make a transformation $\frac{s}{u} = s_1$ where u is the common denominator for all the $\alpha_j, j = 1, \cdots, p$ and $\beta_j, j = 1, \cdots, q$. Under this transformation each coefficient of s_1 in each gamma in (1.9.1) becomes a positive integer. Then we may expand all the gammas by using the multiplication formula for gamma functions. Then the coefficients of s in every gamma becomes ± 1 and then the H-function becomes a G-function. An illustration of this aspect was seen in Example 1.8.3.

Example 1.9.1. Evaluate the following reaction rate integral in physics and write it as an H-function.

$$I(\alpha, a, b, \rho) = \int_0^\infty x^{\alpha-1} \mathrm{e}^{-ax-bx^{-\rho}} \mathrm{d}x.$$

Solution 1.9.1. From (1.8.18) in Example 1.8.3 we have

$$\int_0^\infty x^{\alpha-1} \mathrm{e}^{-ax-bx^{-\rho}} \mathrm{d}x = \frac{a^{-\alpha}}{\rho} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\alpha + s)\Gamma(\frac{s}{\rho})(ab^{1/\rho})^{-s} \mathrm{d}s. \tag{1.9.4}$$

Writing the right side with the help of (1.9.1) we have the following:

$$\int_0^\infty x^{\alpha-1} \mathrm{e}^{-ax-bx^{-\rho}} \mathrm{d}x = \frac{1}{\rho a^\alpha} H_{0,2}^{2,0}\left[ab^{\frac{1}{\rho}}\Big|_{(\alpha,1),(0,\frac{1}{\rho})}\right]. \tag{1.9.5}$$

Example 1.9.2. Let $x_1, \cdots, x_k$ be independently distributed real scalar gamma random variables with the parameters $(\alpha_j, 1), j = 1, \cdots, k$. Let $\gamma_1, \cdots, \gamma_k$ be real constants.
Let

$$u = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_k^{\gamma_k}.$$

Evaluate the density of u.

Solution 1.9.2. Let us take the $(s-1)^{th}$ moment of u or the Mellin transform of the density of u.

$$E(u^{s-1}) = E[x_1^{\gamma_1} \cdots x_k^{\gamma_k}]^{s-1} = E(x_1^{\gamma_1(s-1)}) \cdots E[x_k^{\gamma_k(s-1)}]$$

due to independence. But for a real gamma random variable, with parameters $(\alpha_j, 1)$, the $[\gamma_j(s-1)]^{th}$ moment is the following:

$$E[x_j^{\gamma_j(s-1)}] = \frac{\Gamma(\alpha_j + \gamma_j(s-1))}{\Gamma(\alpha_j)} = \frac{\Gamma(\alpha_j - \gamma_j + \gamma_j s)}{\Gamma(\alpha_j)} \text{ for } \Re(\alpha_j + \gamma_j(s-1)) > 0. \tag{1.9.6}$$

Then

$$E(u^{s-1}) = \prod_{j=1}^{k} \frac{\Gamma(\alpha_j - \gamma_j + \gamma_j s)}{\Gamma(\alpha_j)}.$$

The density of u, denoted by $g(u)$, is available from the inverse Mellin transform. That is,

$$\begin{aligned} g(u) &= \frac{1}{\left\{\prod_{j=1}^{k}\Gamma(\alpha_j)\right\}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left\{\prod_{j=1}^{k} \Gamma(\alpha_j - \gamma_j + \gamma_j s)\right\} u^{-s} \mathrm{d}s \\ &= \begin{cases} \frac{1}{\left\{\prod_{j=1}^{k}\Gamma(\alpha_j)\right\}} H_{0,k}^{k,0}\left[u\big|_{(\alpha_j-\gamma_j,\gamma_j),j=1,\cdots,k}\right], u > 0, \\ 0, \quad \text{elsewhere.} \end{cases} \end{aligned}$$

This is the density function for the product of arbitrary powers of independently distributed real scalar gamma random variables.

By using similar procedures one can obtain and write in terms of H-functions, the densities of products of arbitrary powers of real scalar type-1 beta and type-2 beta random variables or arbitrary powers of products and ratios of real scalar gamma, type-1, type-2 beta or other such positive variables. Some details may by seen from Mathai (1993) and Mathai and Saxena (1978).

Exercises 1.9.

1.9.1. Prove that

$$H_{p,q}^{m,n}[z\big|_{(b_1,\beta_1),\cdots,(b_q,\beta_q)}^{(a_1,\alpha_1),\cdots,(a_p,\alpha_p)}] = H_{q,p}^{n,m}[\frac{1}{z}\big|_{(1-a_1,\alpha_1),\cdots,(1-a_p,\alpha_p)}^{(1-b_1,\beta_1),\cdots,(1-b_q,\beta_q)}].$$

1.9.2. Evaluate the Mellin-Barnes integral

$$E_\alpha(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)}(-z)^{-s}\mathrm{d}s \tag{1.9.7}$$

and show that $E_\alpha(z)$ is the Mittag-Leffler series.

1.9.3. Evaluate the Laplace transform of $E_\alpha(z^\alpha)$ of Exercise 1.9.2, in (1.9.8), with parameter p.

1.9.4. A generalization of Mittag-Leffler function $E_{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r+\beta)}$. Evaluate the Laplace transform of $t^{\beta-1}E_{\alpha,\beta}(z^\alpha)$.

1.9.5. Write $E_\alpha(z)$ as an H-function.

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