CHAPTER 3

FRACTIONAL CALCULUS

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3.0. Introduction

A generalization of Saigo operators of generalized fractional calculus associated with Appell function of the third kind, as the kernel, is presented here. These operators are general in nature and provide extension of the well-known operators of fractional calculus which includes, among others, the Riemann-Liouville , Weyl, Erdélyi-Kober and Saigo operators. Certain properties of these new operators associated with the H-functions are established. Further, we derive the solution of a general integro-differential equation of Volterra-type and discuss its special cases. It is expected that this study will motivate the readers to pursue research in the field of generalized fractional calculus and emerging and potential areas of fractional differential equations.

3.1. Appell Functions

Definitions and properties of the Appell functions are available from the book (Erdélyi et al, 1953).

3.1.1. Appell function F_3

Notation 3.1.1. F_3: Appell function of the fi rst kind

Definition 3.1.1. The function F_3 is defined in the form

$$F_3(a,a',b,b';c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_n(b')_n}{(c)_{m+n}} \frac{x^m y^n}{(m)!(n)!} \tag{3.1.1}$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1\left[{a',b' \atop c+m}; y\right] \frac{x^m}{(m)!} \tag{3.1.2}$$

where $\max\{|x|,|y|\} < 1$.

It reduces to Gauss hypergeometric function as

$${}_2F_1(a,b;c;z) = F_3(a,a',b,b';c;z,0) \tag{3.1.3}$$

$$= F_3(a,0,b,b';c;z,y) \tag{3.1.4}$$

$$= F_3(a,a',b,0;c;z,y) \tag{3.1.5}$$

Note 3.1.1. We see that a and b or a' and b' can be interchanged with each other in $F_3(a,a',b,b';c;x,y)$.

3.1.2. Mellin transform of F_3

Notation 3.1.2. $M\{f(x);s\}, F(s)$, Mellin transform of $f(x)$

Definition 3.1.2.

$$M\{f(x);s\} = F(s) = \int_0^{\infty} x^{s-1} f(x)\mathrm{d}x, (s \in C) \tag{3.1.6}$$

The inversion formula for this transform is given by

$$f(x) = \frac{1}{2\pi i} \int_{\xi - i\infty}^{\xi + i\infty} F(s) x^{-s} \mathrm{d}s, \tag{3.1.7}$$

where line of integration runs parallel to the imaginary axis along the line $\Re(s) = \xi$ from $-\infty$ to ∞.

Lemma 3.1.1. *If,* $\Re(c) > 0, \Re(s) > \max[\Re(-a'), \Re(-b'), \Re(a+b-c)]$, *then from [Prudnikov et al, 1990, eq.(8.4.51.2)], we have*

$$M\left\{(1-x)^{c-1}F_3(a,a',b,b';c;1-x,1-\frac{1}{x});s\right\}$$
$$=\Gamma\begin{bmatrix} c, & s+a', & s+b', & s+c-a-b \\ & s+a'+b', & s+c-a, & s+c-b \end{bmatrix}, \tag{3.1.8}$$

where $\Gamma[{a,b,c,\cdots \atop d,e,f,\cdots}]$ represents the ratio of the product of several gamma functions:

$$\frac{\Gamma(a)\Gamma(b)\Gamma(c)\cdots}{\Gamma(d)\Gamma(e)\Gamma(f)\cdots}=\Gamma\begin{bmatrix} a, & b, & c, & \cdots \\ d, & e, & f, & \cdots \end{bmatrix}.$$

3.1.3. Asymptotic expansion of the Appell function F_3

The following lemma gives the asymptotic estimates of the Appell function F_3.

Lemma 3.1.2. *For $a,a',b,b',c \in C$, and $z \in C$, there holds the following asymptotic expansion for F_3 near $z=0,1,\infty$.*

$$\text{Near } z=0: F_3(a,a',b,b';c;z,\frac{z}{z-1})=O(z^{\min[0,1-\Re(c)]}) \text{ as } (z\to 0); \tag{3.1.9}$$

$$\text{Near } z=1: F_3(a,a',b,b';c;z,\frac{z}{z-1})=O((1-z)^{\min[\Re(a'),\Re(b'),\Re(c-a-b)]}) \text{ as } (z\to 1); \tag{3.1.10}$$

$$\text{Near } z=\infty: F_3(a,a',b,b';c;z,\frac{z}{z-1})=O(z^{\max[\Re(a'),\Re(b'),\Re(c-a-b)]}) \text{ as } (z\to \infty). \tag{3.1.11}$$

Note 3.1.2. The asymptotic estimates (3.1.9),(3.1.10) and (3.1.11) of the Appell function F_3 can be obtained from the result (Prudnikov et al, 1990, p.452, (7.2.4.74)).

Exercises 3.1.

3.1.1. Establish the Lemma 3.1.2.

3.1.2. With the help of Lemma 3.1.2 or otherwise derive the asymptotic estimates for the hypergeometric function ${}_2F_1(a,b;c;z)$, near $z=0,1,\infty$.

3.1.3. For $a, b, c \in C$ with $\Re(c) > 0$ and $z \in C$, prove the following asymptotic relations near $z = 1$:

(i) ${}_2F_1(a, b; c; z) = O(1)(z \to 1-)$ for $\Re(c - a - b) > 0$

(ii) ${}_2F_1(a, b; c; z) = O((1 - z)^{c-a-b})(z \to 1-)$ for $\Re(c - a - b) < 0$

(iii) ${}_2F_1(a, b; c; z) = O(\log(1 - z))(z \to 1-)$ for $\Re(c - a - b) = 0;$ $a, b \neq 0, -1, -2, \cdots$ and $|\arg z| < \pi$.

3.1.4. Prove that

$$F_3(a, a', b, b'; c; 0, y) = F_3(0, a', b, b'; c; x, y) = F_3(a, a', 0, b'; c; x, y) = {}_2F_1(a, b; c; y).$$

3.2. Generalized Fractional Calculus Operators

3.2.1. Saigo-Maeda and Saigo operators

Notation 3.2.1. $(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x)$, Saigo-Maeda (1996)

Notation 3.2.2. $(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x)$, Saigo-Maeda (1996)

Notation 3.2.3. $(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x)$, Saigo-Maeda (1996)

Notation 3.2.4. $(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x)$, Saigo-Maeda (1996)

Notation 3.2.5. $(I_{0+}^{\alpha,\beta,\eta} f)(x)$, Saigo (1978)

Notation 3.2.6. $(I_{-}^{\alpha,\beta,\eta} f)(x)$, Saigo (1978)

Notation 3.2.7. $(D_{0+}^{\alpha,\beta,\eta} f)(x)$, Saigo (1978)

Notation 3.2.8. $(D_{-}^{\alpha,\beta,\eta} f)(x)$, Saigo (1978).

Definition 3.2.1. (Saigo-Maeda,1996)

$$(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}) f(t) \mathrm{d}t; \quad (3.2.1)$$

$(\Re(\gamma) > 0)$

$$(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k (I_{0+}^{\alpha,\alpha',\beta+k,\beta',\gamma+k} f)(x) \tag{3.2.2}$$

$(\Re(\gamma) \leq 0; k = [-\Re(\gamma)] + 1)$;

Definition 3.2.2. (Saigo-Maeda,1996)

$$(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}) f(t)\mathrm{d}t; \tag{3.2.3}$$

$(\Re(\gamma) > 0)$

$$(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^k (I_{0+}^{\alpha,\alpha',\beta,\beta'+k,\gamma+k} f)(x) \tag{3.2.4}$$

$(\Re(\gamma) \leq 0; k = [-\Re(\gamma)] + 1)$;

Definition 3.2.3. (Saigo-Maeda,1996)

$$(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = (I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f)(x) \tag{3.2.5}$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k (I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k} f)(x) \tag{3.2.6}$$

$(\Re(\gamma) > 0; k = [\Re(\gamma)] + 1)$;

Definition 3.2.4. (Saigo-Maeda,1996)

$$(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = (I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f)(x) \tag{3.2.7}$$

$$= \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^k (I_{0+}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k} f)(x) \tag{3.2.8}$$

$(\Re(\gamma) > 0; k = [\Re(\gamma)] + 1)$.

Definition 3.2.5. (Saigo,1978)

$$(I_{0+}^{\alpha,\beta,\eta} f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}) f(t) \mathrm{d}t, (\Re(\alpha) > 0) \tag{3.2.9}$$

$$(I_{0+}^{\alpha,\beta,\eta} f)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k (I_{0+}^{\alpha+k,\beta-k,\eta-k} f)(x), (\Re(\alpha) \leq 0; k = [\Re(-\alpha)] + 1); \tag{3.2.10}$$

Definition 3.2.6. (Saigo,1978)

$$(I_{-}^{\alpha,\beta,\eta} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}) f(t) \mathrm{d}t, (\Re(\alpha) > 0); \tag{3.2.11}$$

$$(I_{-}^{\alpha,\beta,\eta} f)(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^k (I_{0+}^{\alpha+k,\beta-k,\eta} f)(x), (\Re(\alpha) \leq 0; k = [\Re(-\alpha)] + 1); \tag{3.2.12}$$

Definition 3.2.7. (Saigo,1978)

$$(D_{0+}^{\alpha,\beta,\eta} f)(x) = (I_{0+}^{-\alpha,-\beta,\alpha+\eta} f)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k (I_{0+}^{-\alpha+k,-\beta-k,\alpha+\eta-k} f)(x) \tag{3.2.13}$$
$$(\Re(\alpha) > 0; k = [\Re(-\alpha)] + 1);$$

Definition 3.2.8. (Saigo,1978)

$$(D_{-}^{\alpha,\beta,\eta} f)(x) = (I_{-}^{-\alpha,-\beta,\alpha+\eta} f)(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^k (I_{-}^{-\alpha+k,-\beta-k,\alpha+\eta} f)(x) \tag{3.2.14}$$
$$(\Re(\alpha) > 0; k = [\Re(\alpha)] + 1).$$

Theorem 3.2.1. *The Saigo-Maeda operators reduce to that of Saigo operators by virtue of the following identities:*

$$(I_{0+}^{\alpha,0,\beta,\beta',\gamma} f)(x) = (I_{0+}^{\gamma,\alpha-\gamma,-\beta} f)(x) \quad (\gamma \in C); \tag{3.2.15}$$

$$(I_{-}^{\alpha,0,\beta,\beta',\gamma} f)(x) = (I_{-}^{\gamma,\alpha-\gamma,-\beta} f)(x) \quad (\gamma \in C); \tag{3.2.16}$$

$$(D_{0+}^{0,\alpha',\beta,\beta',\gamma} f)(x) = (D_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f)(x) \quad (\Re(\gamma) > 0); \tag{3.2.17}$$

$$(D_{0+}^{0,\alpha',\beta,\beta',\gamma} f)(x) = (D_{0+}^{\gamma,\alpha'-\gamma,-\beta'-\gamma} f)(x) \quad (\Re(\gamma) > 0). \tag{3.2.18}$$

Theorem 3.2.2. *The Saigo-Maeda operators obey the formula of fractional integration by parts:*

$$\int_0^\infty (I_{+}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) g(x) \mathrm{d}x = \int_0^\infty (I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} g)(x) f(x) \mathrm{d}x. \tag{3.2.19}$$

Example 3.2.1. Derive the Saigo-Maeda operators of power functions :

$$(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1})(x) = \Gamma\begin{bmatrix} \rho, & \rho+\gamma-\alpha-\alpha'-\beta, & \rho+\beta'-\alpha' \\ \rho+\gamma-\alpha-\alpha', & \rho+\gamma-\alpha'-\beta, & \rho+\beta' \end{bmatrix} x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{3.2.20}$$

where

$$\Re(\gamma) > 0, \Re(\rho) > \max[0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')],$$

and

$$(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1})(x)$$
$$= \Gamma\begin{bmatrix} 1+\alpha+\alpha'-\gamma-\rho, & 1+\alpha+\beta'-\gamma-\rho, & 1-\beta-\rho \\ 1-\rho, & 1+\alpha+\alpha'+\beta'-\gamma-\rho, & 1+\alpha-\beta-\rho \end{bmatrix} x^{\rho-\alpha-\alpha'+\gamma-1} \tag{3.2.21}$$

where $\Re(\gamma) > 0, \Re(\rho) < 1 + \min[\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)]$.

These results can be derived from the integral (3.1.8).

Remark 3.2.1. The inverses of the operators defined by (3.1.1) and (3.1.3) are given by (Saigo et al , 1996) as

$$(I_{+}^{\alpha,\alpha',\beta,\beta',\gamma})^{-1} = (I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}), \tag{3.2.22}$$

and

$$(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma})^{-1} = (I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}). \tag{3.2.23}$$

Exercises 3.2.

3.2.1. Prove Theorem 3.2.2.

3.2.2. Complete the proof of Example 3.2.1.

3.2.3. Derive the Saigo operators $I_{0+}^{\alpha,\beta,\eta}$ and $I_{-}^{\alpha,\beta,\eta}$ of the power functions.

3.2.4. Find the value of

$$(I_{+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} {}_2F_1(a,b,c;ex))(x)$$

and give the condition of its validity.

3.2.5. Find the value of

$$(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} {}_2F_1(a,b,c;\frac{e}{x}))(x)$$

and give the conditions of its validity.

3.3. Generalized Fractional Calculus of the H-function

Notation 3.3.1. $H_{p,q}^{m,n}(z), H_{p,q}^{m,n}\left[z\Big|_{(a_p,A_p)}^{(b_q,B_q)}\right], H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\cdots,(b_q,B_q)}^{(a_1,A_1),\cdots,(a_p,A_p)}\right]$

Definition 3.3.1. The H-function is defined by means of a Mellin-Barnes type integral in the following manner (Mathai and Saxena,1978):

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]$$
$$= H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\cdots,(b_q,B_q)}^{(a_1,A_1),\cdots,(a_p,A_p)}\right] = \frac{1}{2\pi i}\int_L \Theta(\xi) z^{-\xi} \mathrm{d}\xi, (i = (-1)^{1/2}) \qquad (3.3.1)$$

where

$$\Theta(\xi) = \frac{\left\lfloor \prod_{j=1}^{m} \Gamma(b_j + B_j\xi)\right\rfloor \left\lfloor \prod_{j=1}^{n} \Gamma(1 - a_j - A_j\xi)\right\rfloor}{\left\lfloor \prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j\xi)\right\rfloor \left\lfloor \prod_{j=n+1}^{p} \Gamma(a_j + A_j\xi)\right\rfloor}, \qquad (3.3.2)$$

and an empty product is always interpreted as unity; $m, n, p, q \in N_0$; $0 \le m \le q$, $1 \le m \le p$; $A_i, B_j \in R_+, a_i, b_j \in R$ or $C(i = 1, \cdots, p;\ j = 1, \cdots, q)$ such that

$A_i(b_j+k) \neq B_j(a_i-\lambda-1)(k,\lambda \in N_0; (i=1,\cdots n; j=1,\cdots,m))$. The contour L is either $L_{-\infty}, L_{+\infty}$ or $L_{i\tau\infty}$. These contours are defined explicitly in the monograph by Prudnikov et al (1990) and Mathai (1993). A detailed and comprehensive account of the H-function is available from the monographs written by Mathai and Saxena (1978), Prudnikov et al (1990) and Kilbas and Saigo (2004). In what follows $L = L_{i\tau\infty}$ is a contour starting at the points $\tau - i\infty$ and terminating at the point $\tau + i\infty$ with $\tau \in \Re$. The H-function makes sense, if

$$\text{(i)}\ \ \Omega = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0; |\arg z| < \frac{1}{2}\pi\Omega; z \neq 0. \qquad (3.3.3)$$

$$\text{(ii)}\ \ \Omega = 0,\ \Delta\tau + \Re(\mu) < -1,\ \arg z = 0,\ z \neq 0,$$

where

$$\Delta = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j; \mu = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2}. \qquad (3.3.4)$$

The following properties of the H-function will be required in the proof of the results that follows: For $\rho, a \in C, \sigma \in \Re_+$, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \left\{x^{\rho-1} H_{p,q}^{m,n}\left[ax^{\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\}$$

$$= x^{\rho-k-1} H_{p+1,q+1}^{m,n+1}\left[ax^{\sigma}\Big|_{(b_q,B_q),(1-\rho+k,\sigma)}^{(1-\rho,\sigma),(a_p,A_p)}\right], \qquad (3.3.5)$$

$$= (-1)^k x^{\rho-k-1} H_{p+1,q+1}^{m+1,n}\left[ax^{\sigma}\Big|_{(1-\rho+k,\sigma),(b_q,B_q),}^{(a_p,A_p),(1-\rho,\sigma)}\right]. \qquad (3.3.6)$$

3.3.1. Left-sided generalized fractional integration of the H-function

In what follows,

$$K = \max_{1\leq j\leq n}\left[\frac{1-\Re(a_j)}{A_j}\right] \ \text{and}\ K^* = \min_{1\leq j\leq m}\left[\frac{-\Re(b_j)}{B_j}\right]. \qquad (3.3.7)$$

Theorem 3.3.1. *Let* $\alpha, \alpha', \beta, \beta', \gamma \in C, \Re(\gamma) > 0$. *Further let the constants* $a_i, b_j \in \Re_+, (i=1,\cdots,p; j=1,\cdots,q), \rho \in C, \sigma \in \Re_+, |\arg a| < \pi\Omega/2$ *be given and satisfy the condition* $\sigma\max[\tau, K^*] < \Re(\rho) + \min[0, \Re(\beta'-\alpha'), \Re(\gamma-\alpha-\beta-\alpha')]$.

Then the generalized fractional integral $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ *of the H-function exists and the following relation holds:*

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}H_{p,q}^{m,n}\left[at^{\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right)(x)$$
$$= x^{\rho+\gamma-\alpha-\alpha'-1}H_{p+3,q+3}^{m,n+3}\left[ax^{\sigma}\Big|_{(b_q,B_q),(1-\rho-\beta',\sigma),(1+\alpha+\alpha'-\gamma-\rho,\sigma),(1+\alpha'+\beta-\gamma-\rho,\sigma)}^{(1-\rho,\sigma),(1+\alpha'-\beta'-\rho,\sigma),(1+\alpha+\alpha'+\beta-\gamma-\rho,\sigma),(a_p,A_p)}\right]. \tag{3.3.8}$$

Proof 3.3.1. To prove (3.3.8), express the H-function in terms of its Mellin-Barnes contour, interchange the order of integration and apply the power function formula (3.2.20).

Corollary 3.3.1. *Let* $\alpha,\beta,\eta \in C, \Re(\alpha) > 0$. *Further let the constants* $a_i, b_j \in C, A_i, B_j \in \Re_+, (i = 1,\cdots,p;\ j = 1,\cdots,q), \rho \in C, \sigma \in \Re_+, |\arg a| < \pi\Omega/2$ *be given and satisfy the condition* $\sigma\max[\tau, K^*] < \Re(\rho) + \min[0, \Re(\eta-\beta)]$. *Then the Saigo operator* $I_{0+}^{\alpha,\beta,\eta}$ *of the H-function exists and there holds the formula :*

$$\left(I_{0+}^{\alpha,\beta,\eta}t^{\rho-1}H_{p,q}^{m,n}\left[at^{\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right)(x)$$
$$= x^{\rho-\beta-1}H_{p+2,q+2}^{m,n+2}\left[ax^{\sigma}\Big|_{(b_q,B_q),(1+\beta-\rho,\sigma),(1-\alpha-\eta-\rho,\sigma)}^{(1-\rho,\sigma),(1+\beta-\eta-\rho,\sigma),(a_p,A_p)}\right]. \tag{3.3.9}$$

Hint: Use the identity (3.2.15).

3.3.2. Right-sided generalized fractional integration of the H-function

Theorem 3.3.2. *Let* $\alpha,\alpha',\beta,\beta',\gamma \in C, \Re(\gamma) > 0$. *Further let the constants* $a_i, b_j \in \Re_+ (i = 1,\cdots,p;\ j = 1,\cdots,q), \rho \in C, \sigma \in \Re_+, |\arg a| < \pi\Omega/2$ *be given and satisfy the condition* $\sigma\min[\tau, K] + 1 > \Re(\rho) + \max[\Re(\gamma-\alpha-\alpha'), \Re(\gamma-\alpha-\beta'), \Re(\beta-\alpha')]$. *Then the generalized fractional integral* $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ *of the H-function exists and the following relation holds:*

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}H_{p,q}^{m,n}\left[at^{\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right)(x)$$
$$= x^{\rho+\gamma-\alpha-\alpha'-1}$$
$$\times H_{p+3,q+3}^{m+3,n}\left[ax^{\sigma}\Big|_{(1-\rho-\beta,\sigma),(1+\alpha+\beta'-\gamma-\rho,\sigma),(1+\alpha+\alpha'-\gamma-\rho,\sigma),(b_q,B_q)}^{(a_p,A_p),(1-\rho,\sigma),(1+\alpha-\beta-\rho,\sigma),(1+\alpha+\alpha'+\beta'-\gamma-\rho,\sigma),}\right]. \tag{3.3.10}$$

Proof 3.3.2. It is similar to the proof of Theorem 3.3.1.

Corollary 3.3.2. *Let $\alpha, \beta, \eta \in C, \Re(\alpha) > 0$. Further let the constants $a_i, b_j \in C$, $A_i, B_j \in \mathcal{R}_+, (i = 1, \cdots, p; j = 1, \cdots, q), \rho \in C, \sigma \in \mathcal{R}_+$, $|\arg a| < \pi\Omega/2$ be given and satisfy the condition $\sigma \min[\tau, K] + 1 > \Re(\rho) - \min[\Re(\beta), \Re(\gamma)]$. Then the Saigo operator $I_-^{\alpha,\beta,\eta}$ of the H-function exists and there holds the formula :*

$$\left(I_-^{\alpha,\beta,\eta} t^{\rho-1} H_{p,q}^{m,n}\left[at^{\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right)(x) = x^{\rho-\beta-1} H_{p+2,q+2}^{m+2,n}\left[ax^{\sigma}\Big|_{(1+\beta-\rho,\sigma),(1+\eta-\rho,\sigma),(b_q,B_q)}^{(a_p,A_p),(1-\rho,\sigma),(1+\alpha+\beta+\eta-\rho,\sigma)}\right]. \tag{3.3.11}$$

3.3.3. Left-sided generalized fractional differentiation of the H-function

In this section left-sided generalized fractional derivative $D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the H-function is investigated. Following a similar procedure, we establish the following result:

Theorem 3.3.3. *Let $\alpha, \alpha', \beta, \beta', \gamma \in C, \Re(\gamma) > 0$. Further let the constants $a_i, b_j \in \Re_+(i = 1, \cdots, p; j = 1, \cdots, q), \rho \in C, \sigma \in \Re_+$, $|\arg a| < \pi\Omega/2$ be given and satisfy the condition $\sigma \max[\tau, K^*] < \Re(\rho) + \min[0, \Re(\alpha - \beta), \Re(\alpha' + \beta' + \alpha - \gamma)]$. Then the generalized fractional derivative $D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the H-function exists and the following relation holds:*

$$\begin{aligned}&\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} H_{p,q}^{m,n}\left[at^{\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right)(x)\\ &= x^{\rho+\alpha+\alpha'-\gamma-1}\\ &\times H_{p+3,q+3}^{m,n+3}\left[ax^{\sigma}\Big|_{(b_q,B_q),(1+\beta-\rho,\sigma),(1-\alpha-\beta'+\gamma-\rho,\sigma),(1-\alpha-\alpha'+\gamma-\rho,\sigma)}^{(1-\rho,\sigma),(1-\alpha+\beta-\rho,\sigma),(1-\alpha-\alpha'-\beta'+\gamma-\rho,\sigma),(a_p,A_p)}\right],\end{aligned} \tag{3.3.12}$$

Corollary 3.3.3. *Let $\alpha, \beta, \eta, \gamma \in C, \Re(\alpha) > 0$. Further let the constants $a_i, b_j \in C$, $A_i, B_j \in \Re_+(i = 1, \cdots, p; j = 1, \cdots, q), \rho \in C, \sigma \in \Re_+$, $|\arg a| < \pi\Omega/2$ be given and satisfy the condition $\sigma \max[\tau, K^*] < \Re(\rho) + \min[0, \Re(\alpha + \beta + \eta)]$. Then the Saigo operator $D_{0+}^{\alpha,\beta,\eta}$ of the H-function exists and there holds the formula*

$$\left(D_{0+}^{\alpha,\beta,\eta} t^{\rho-1} H_{p,q}^{m,n}\left[at^{\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right)(x)$$
$$= x^{\rho-\beta-1} H_{p+2,q+2}^{m,n+2}\left[ax^{\sigma}\Big|_{(b_q,B_q),(1-\beta-\rho,\sigma),(1-\eta-\rho,\sigma)}^{(1-\rho,\sigma),(1-\alpha-\beta-\eta-\rho,\sigma),(a_p,A_p)}\right]. \quad (3.3.13)$$

3.3.4. Right-sided generalized fractional differentiation of the H-function

Theorem 3.3.4. *Let $\alpha, \alpha', \beta, \beta', \gamma \in C, \Re(\gamma) > 0$. Further let the constants $a_i, b_j \in \Re_+ (i = 1, \cdots, p; j = 1, \cdots, q), \rho \in C, \sigma \in \Re_+, |\arg a| < \pi\Omega/2$ be given and satisfy the condition $\sigma \max[\tau, K] + 1 > \Re(\rho) - \min[0, \Re(\gamma - \alpha - \alpha' - k), \Re(\gamma - \alpha' - \beta), \Re(\beta')]$. Then the generalized fractional derivative $D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the H-function exists and the following relation holds:*

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} H_{p,q}^{m,n}\left[at^{\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right)(x)$$
$$= (-1)^{[\Re(\gamma)]+1} x^{\rho+\alpha+\alpha'-\gamma-1}$$
$$\times H_{p+3,q+3}^{m=3,n}\left[ax^{\sigma}\Big|_{(1+\beta'-\rho,\sigma),(1-\alpha'-\beta+\gamma-\rho,\sigma),(1-\alpha-\alpha'+\gamma-\rho,\sigma),(b_q,B_q)}^{(a_p,A_p),(1-\rho,\sigma),(1-\alpha'+\beta'-\rho,\sigma),(1-\alpha-\alpha'-\beta'+\gamma-\rho,\sigma)}\right]. \quad (3.3.14)$$

Proof 3.3.3. It is similar to the proof of Theorem 3.3.1.

Corollary 3.3.4. *Let $\alpha, \beta, \eta \in C, \Re(\alpha) > 0$. Further let the constants $a_i, b_j \in C$, $A_i, B_j \in \Re_+, (i = 1, \cdots, p; j = 1, \cdots, q), \rho \in C, \sigma \in \Re_+, |\arg a| < \pi\Omega/2$ be given and satisfy the condition $\sigma \min[\tau, K]+1 > \Re(\rho)+\max[\Re(\beta)+[\Re(\alpha)]+1, -\Re(\alpha+\eta)]$. Then the Saigo operator $D_{-}^{\alpha,\beta,\eta}$ of the H-function exists and there holds the formula*

$$\left(D_{-}^{\alpha,\beta,\eta} t^{\rho-1} H_{p,q}^{m,n}\left[at^{\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right)(x)$$
$$= (-1)^{[\Re(\alpha)]+1} x^{\rho+\beta-1} H_{p+2,q+2}^{m+2,n}\left[ax^{\sigma}\Big|_{(1-\beta-\rho,\sigma),(1+\alpha+\eta-\rho,\sigma),(b_q,B_q),}^{(a_p,A_p),(1-\rho,\sigma),(1-\beta+\eta-\rho,\sigma)}\right]. \quad (3.3.15)$$

3.3.5. Mellin transform of Saigo-Maeda operators

Let $L_p(\Re_+)$ be the usual Lebesgue class on $\Re_+$ with $1 \le p < \infty$. We define $M_p(\Re_+)$ the class of all functions of $f \in L_p(\Re_+)$ with $p > 2$ which are inverse Mellin transforms of functions in $L_q(\Re_+)$, where $q = p/(p-1)$.

Theorem 3.3.5. *Let* $1 \leq p \leq 2$ *and the constants* $\alpha, \alpha', \beta, \beta', \gamma \in C$, *with* $\Re(\gamma) > 0$ *satisfy* $\Re(s) < 1 + \min[0, \Re(\beta' - \alpha'), \Re(\gamma - \alpha - \alpha' - \beta)]$, *then for* $f \in L_p(\Re_+)$ *(or* $f \in M_p(\Re_+)$ *with* $p > 2$ *) the following formula holds:*

$$M\left\{x^{\alpha+\alpha'-\gamma}\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)\right\}(s) = \Gamma\begin{bmatrix} 1-s, & 1-\alpha'+\beta'-s, & 1-\alpha-\alpha'-\beta+\gamma-s \\ 1+\beta'-s, & 1-\alpha-\alpha'+\gamma-s, & 1-\alpha'-\beta+\gamma-s \end{bmatrix} M[f(x)](s). \tag{3.3.16}$$

Proof 3.3.4. Interchange the order of integration and apply the formula (3.2.21), the result follows.

Corollary 3.3.5. *If* $\Re(\alpha) > 0$ *and* $\Re(s) < 1 + \min[0, \eta - \beta]$, *then there holds the following relation for* $f(x) \in L_p(0, \infty)$ *with* $1 \leq p \leq 2$ *or* $f(x) \in M_p(0, \infty)$ *with* $p > 2$ *:*

$$M\left\{x^{\beta} I_{0+}^{\alpha,\beta,\eta} f\right\}(x) = \frac{\Gamma(1-s)\Gamma(\eta-\beta+1-s)}{\Gamma(1-\beta-s)\Gamma(\alpha+\eta+1-s)} M\{f(x)\}. \tag{3.3.17}$$

Following a similar method, we establish

Theorem 3.3.6. *Let* $1 \leq p \leq 2$ *and the constants* $\alpha, \alpha', \beta, \beta', \gamma \in C$, *with* $\Re(\gamma) > 0$ *satisfy* $\Re(s) > \max[\Re(-\alpha - \alpha' + \gamma), \Re(-\alpha - \beta' + \gamma), \Re(\beta)]$, *then for* $f \in L_p(\Re_+)$ *(or* $f \in M_p(\Re_+)$ *with* $p > 2$ *) the following relation holds:*

$$M\left\{x^{\alpha+\alpha'-\gamma}\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)\right\}(s) = \Gamma\begin{bmatrix} s+\alpha+\alpha'-\gamma, & s+\alpha+\beta'-\gamma, & s-\beta \\ s+\alpha+\alpha'+\beta'-\gamma, & s, & s+\alpha-\beta \end{bmatrix} M[f(x)](s). \tag{3.3.18}$$

Corollary 3.3.6. *If* $\Re(\alpha) > 0$ *and* $\Re(s) > -\min[\Re(\beta), \Re(\eta)]$, *then there holds the following formula for* $f(x) \in L_p(0, \infty)$ *with* $1 \leq p \leq 2$ *or* $f(x) \in M_p(0, \infty)$ *with* $p > 2$ *:*

$$M\left\{x^{\beta} I_{0+}^{\alpha,\beta,\eta} f\right\}(x) = \frac{\Gamma(\beta+s)\Gamma(\eta+s)}{\Gamma(s)\Gamma(\alpha+\beta+\eta+s)} M\{f(x)\}. \tag{3.3.19}$$

Exercises 3.3.

3.3.1. Establish Theorem 3.3.3.

3.3.2. Establish Theorem 3.3.4.

3.3.3. Establish Theorem 3.3.5.

3.3.4. Establish Theorem 3.3.6.

3.3.5. Establish Corollary 3.3.5.without using Theorem 3.3.5.

3.3.6. Establish Corollary 3.3.6.without using Theorem 3.3.6.

3.3.7. Establish Theorem 3.3.1 for $\Re(\gamma) \leq 0$ and hence derive the corresponding theorem for the Saigo operator $I_{0+}^{\alpha,\beta,\eta}$.

3.3.8. Establish Theorem 3.3.2 for $\Re(\gamma) \leq 0$ and hence derive the corresponding theorem for the Saigo operator $I_{-}^{\alpha,\beta,\eta}$.

3.3.9. Evaluate the integral

$$\int_0^x t^{\rho-1}(x-t)^{\sigma-1} H_{p,q}^{m,n}\left(at^{\lambda}(x-t)^{\mu}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right) \mathrm{d}t, (\lambda, \mu > 0) \tag{3.3.20}$$

and give the conditions of its validity. Also find the value of

$$I_{0+}^{\alpha}\left(t^{\rho-1} H_{p,q}^{m,n}\left[at^{\lambda}(x-t)^{\mu}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right)(x).$$

3.3.6. A general class of multivariable polynomials

Notation 3.3.2. $S_L^{h_1,\cdots,h_s}(x_1,\cdots,x_s)$, multivariable polynomials (Srivastava et al,1987).

Notation 3.3.3. $S_\lambda^h(x)$ a general class of polynomials (Srivastava, 1972).

Definition 3.3.2.

$$S_L^{h_1,\cdots,h_s}(x_1,\cdots,x_s) = \sum_{k_1,\cdots,k_s=0}^{h_1k_1+\cdots+h_sk_s\le L} (-L)_{h_1s_1+\cdots+h_sk_s} A(L;k_1,\cdots,k_s)\frac{x_1^{k_1}}{(k_1)!}\cdots\frac{x_s^{k_s}}{(k_s)!},$$

$$(h_j \in N;\ j=1,\cdots,s) \quad (3.3.21)$$

where $h_1,\cdots,h_s$ are arbitrary positive integers and the coefficients $A(L;k_1,\cdots,k_s)$, $(L,k_j \in N_0;\ j=1,\cdots,s)$ are arbitrary constants, real or complex.

Definition 3.3.3.

$$S_\lambda^h(x) = \sum_{k=0}^{[\lambda/h]} \frac{(-\lambda)_{hk}}{(k)!} A_{\lambda,k}x^k,\ \ \lambda \in N_0 = \{0,1,2,\cdots\}, \quad (3.3.22)$$

where h is an arbitrary positive integer, and the coefficients $A_{\lambda,k}, (\lambda,k \in N_0)$ are arbitrary constants, real or complex.

Note 3.3.1. We note that for $s=1$, (3.3.21) reduces to (3.3.22). By giving suitable values to the coefficients $A_{\lambda,k}$, the polynomials $S_\lambda^h(x)$ yields a number of known polynomials, as special cases. These include, among others, certain orthogonal and non-orthogonal polynomials such as, the Laguerre polynomials, the Hermite polynomials, the Jacobi polynomials, the Bessel polynomials, the Gould-Hopper polynomials, the Brafman polynomials and several others. For the relations of theses polynomials with $S_\lambda^h(x)$, see (Srivastava et al,1983).

Exercises 3.3.

3.3.10. Show that (Saxena et al, 2002a)

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} H_{p,q}^{m,n}(\lambda t^\sigma) S_L^{h_1,\cdots,h_s}[y_1 t^{\lambda_1},\cdots,y_s t^{\lambda_s}]\right)(x)$$

$$= x^{\rho+c-a-a'-1} \sum_{k_1,\cdots,k_s=0}^{h_1k_1+\cdots+h_sk_s\le L} (-L)_{h_1s_1+\cdots+h_sk_s} A(L;k_1,\cdots,k_s)\frac{y_1^{k_1}}{(k_1)!}\cdots\frac{y_s^{k_s}}{(k_s)!} x^{k_1\lambda_1+\cdots+k_s\lambda_s}$$

$$\times H_{p+3,q+3}^{m,n+3}\left[\lambda x^\sigma\Big|_{(b_q,B_q),(\theta+a+a'-c,\sigma),(\theta+a'+b-c,\sigma),(\theta-b',\sigma)}^{(\theta+a+a'+b-c,\sigma),(\theta,\sigma),(\theta+a'-b',\sigma)(a_p,A_p)}\right], \quad (3.3.23)$$

where $\theta = 1-\rho-\sum_{j=1}^s \lambda_j k_j$; $|\arg\lambda| < (\pi/2)\Omega, \Omega > 0, \Re(c) > 0, \sigma > 0; \omega \max[\tau, K^*] < \Re(\rho)+\min[0, \Re(b'-a'), \Re(-a-b-b')]$ and Ω is defined in (3.3.3) and K^* in (3.3.7).

3.3.11. Show that (Saxena et al, 2002a)

$$\Big(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} H_{p,q}^{m,n}(\lambda t^{\sigma}) S_{L}^{h_1,\cdots,h_s}[y_1 t^{\lambda_1},\cdots,y_s t^{\lambda_s}]\Big)(x)$$

$$= x^{\rho+c-a-a'-1} \sum_{k_1,\cdots,k_s=0}^{h_1k_1+\cdots+h_sk_s\le L} (-L)_{h_1s_1+\cdots+h_sk_s} A(L;k_1,\cdots,k_s) \frac{y_1^{k_1}}{(k_1)!}\cdots\frac{y_s^{k_s}}{(k_s)!} x^{k_1\lambda_1+\cdots+k_s\lambda_s}$$

$$\times H_{p+3,q+3}^{m+3,n}\left[\lambda x^{\sigma}\Big|_{(\theta+a+a'-c,\sigma),(\theta+a+b'-c,\sigma),(\theta-b,\sigma),(b_q,B_q)}^{(a_p,A_p),(\theta+a+a'+b'-c,\sigma),(\theta,\sigma),(\theta+a-b,\sigma)}\right], \tag{3.3.24}$$

where $\Re(c) > 0, \sigma\max[\tau,K]+1 > \Re(\rho)+\max[\Re(c-a-a'),\Re(c-a-b'),\Re(b)]$, and K is defined in (3.3.7).

3.3.12. Show that

$$I_{0+}^{\alpha}\left\{t^{\rho-1}H_{p,q}^{m,n}\left[\lambda t^{u}(x-t)^{w}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] S_{N}^{M}\{gt^{\mu}(1-t)^{\nu}\} S_{N'}^{M'}\{g't^{\mu'}(1-t)^{\nu'}\}\right\}$$

$$= \frac{\lambda^{\rho+\sigma-1}}{\Gamma(\sigma)} \sum_{k=0}^{[N/M]}\sum_{k'=0}^{[N'/M']} \frac{(-N)_{MK}(-N')_{M'K'}}{(K!)(K'!)} A_{N,K}A_{N',K'} g^{K} g'^{K'} x^{(\mu+\nu)K+(\mu'+\nu')K'}$$

$$\times H_{p+3,q+1}^{m,n+1}\left[\lambda x^{u+w}\Big|_{(b_q,B_q),(1-\rho-\sigma-\mu K-\nu K-\mu'K'-\nu'K',u+w)}^{(1-\rho-\mu K-\mu'K',u),(1-\sigma-\nu K-\nu'K',w),(a_p,A_P)}\right], \tag{3.3.25}$$

where $u,w,\mu,\nu,\mu',\nu' > 0; \Re(\sigma) > 0$ and give the conditions of its validity. Here $S_N^M(x)$ are the general class of polynomials defined by (3.3.22).

3.4. Fractional Differential Equations

This section deals with the investigation of the solution of a fractional differintegral equation of Volterra-type associated with a confluent hypergeometric function of two variables.

3.4.1. Cauchy-type problem involving a general fractional integro-differential equation of Volterra type

Notation 3.4.1. $E_\alpha(z)$, Mittag- Leffler (1903, 1905) function

Notation 3.4.2. $E_{\alpha,\beta}(z)$, generalized Mittag- Leffler function

Notation 3.4.3. $E_{\beta,\gamma}^{\delta}(z)$, generalized Mittag-Leffler (Prabhakar, 1971)

Notation 3.4.4. $\Phi_3(\beta, \gamma; x, y)$, confluent hypergeometric function of two variables

Notation 3.4.5. $\Phi(\alpha, \beta; z)$, confluent hypergeometric function of one variable

Notation 3.4.6. I_x^α, Riemann-Liouville operator of fractional integration of order α

Notation 3.4.7. D_x^α, Riemann-Liouville operator of fractional differentiation of order α.

Definition 3.4.1.

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, (\alpha \in C, \Re(\alpha) > 0). \quad (3.4.1)$$

Definition 3.4.2.

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, (\alpha, \beta \in C, \Re(\alpha) > 0, \Re(\beta) > 0). \quad (3.4.2)$$

Definition 3.4.3.

$$E_{\beta,\gamma}^{\delta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n z^n}{\Gamma(\beta n + \gamma)(n)!}, (\beta, \gamma, \delta \in C, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0). \quad (3.4.3)$$

Definition 3.4.4.

$$\Phi_3(\beta, \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{(m)!} \frac{y^n}{(n)!}, |x| < \infty, |y| < \infty. \quad (3.4.4)$$

Definition 3.4.5.

$$\Phi(\alpha, \beta; x) = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\beta)_m} x^m, |x| < \infty. \quad (3.4.5)$$

Definition 3.4.6.

$$I_x^\alpha g(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - \xi)^{\alpha - 1} g(\xi) d\xi, \Re(\alpha) > 0. \quad (3.4.6)$$

Definition 3.4.7.

$$D_x^\alpha[g(x)] = I_x^{-\alpha}[g(x)] = D^n I_x^{n-\alpha}[g(x)], n > \Re(\alpha), n \in N^+. \quad (3.4.7)$$

where D_x^n denotes the usual n-fold differentiation, such that $\alpha + n > 0$.

In this section, it is proposed to continue our study of fractional differential equations in continuation of earlier results derived for the fractional relaxation and fractional diffusion problems. We begin this section by proving a general theorem for the fractional differintegral equation, which depends on the following lemma:

Lemma 3.4.1. *If* $\Re(\gamma) > 0, \Re(s) > \max[0, \Re(x), \Re(y)]$, *then the following result holds:*

$$L\{t^{\gamma-1}\Phi_3(\beta,\gamma; xt, yt); s\} = \Gamma(\gamma)s^{-\gamma}(1-\frac{x}{s})^{-\beta}\exp(y/s) \tag{3.4.8}$$

Proof 3.4.1. We have

$$\begin{aligned}\int_0^\infty t^{\gamma-1}e^{-st}\Phi_3(\beta,\gamma; xt, yt)dt &= \int_0^\infty t^{\gamma-1}e^{-st}\sum_{m,n=0}^{\infty}\frac{(\beta)_m x^m t^m y^n t^n}{(\gamma)_{m+n}(m)!(n)!}dt\\ &= \sum_{m,n=0}^{\infty}\frac{(\beta)_m x^m y^n}{(\gamma)_{m+n}(m)!(n)!}\int_0^\infty e^{-st}t^{\gamma+m+n-1}dt\\ &= \frac{\Gamma(\gamma)}{s^\gamma}\sum_{m=0}^{\infty}\frac{(\beta)_m(x/s)^m}{(m)!}\sum_{n=0}^{\infty}\frac{(y/s)^n}{(n)!},\end{aligned}$$

which is equivalent to (3.4.8).

Remark 3.4.1. When $y \to 0$, then by virtue of the identity

$$\lim_{y\to 0}\Phi_3(\beta,\gamma; xt, yt) = \Phi(\beta,\gamma; xt), \tag{3.4.9}$$

we arrive at a known result (Erdélyi et al 1953, p. 270, eq. (6.10.6)):

$$L\{t^{\gamma-1}\Phi_3(\beta,\gamma; xt); s\} = \Gamma(\gamma)s^{-\gamma}(1-\frac{x}{s})^{-\beta}, \tag{3.4.10}$$

where $\Re(\gamma) > 0, \Re(s) > \max[0, \Re(x)]$ and

$$L\{f(t); s\} = F(s) = \int_0^\infty e^{-st}f(t)dt, \Re(s) > 0. \tag{3.4.11}$$

Theorem 3.4.1. *Let* $\beta,\gamma,\lambda,\mu \in C, \nu,\omega \in R, \Re(\alpha) > 0, \Re(\gamma) > 0$, *and* $f(t)$ *is assumed to be continuous on every finite interval* $[0, T], 0 < T < \infty$, *and of the* *exponential order* $e^{\sigma\tau}$ *when* $\tau \to \infty$. *Then for the Cauchy-type problem for the fractional integro-differential equation of Volterra type*

$$D_\tau^\alpha h(\tau) = \frac{\lambda}{\Gamma(\gamma)}\int_0^\tau t^{\gamma-1}\Phi_3(\beta,\gamma;\nu t,\omega t)h(\tau-t)\mathrm{d}t + \mu f(\tau), (0 \le \tau \le 1) \qquad (3.4.12)$$

together with the initial conditions

$$D_\tau^{\alpha-r}h(\tau)|_{\tau=0} = b_r, r = 1,\cdots,n = -[-\Re(\alpha)], (n-1 < \alpha \le n); n \in N, \qquad (3.4.13)$$

where $b_1,\cdots,b_r \in \Re$*, there exists a unique continuous solution given by*

$$h(\tau) = \sum_{r=1}^{n} b_r y_r(\tau) + \mu\int_0^\tau \Theta(\tau-t)f(t)\mathrm{d}t \qquad (3.4.14)$$

where

$$y_r(x) = \sum_{m=0}^{\infty}\frac{\lambda^m x^{\alpha+(\alpha+\gamma)m-r}\Phi_3(\beta m,\alpha+(\alpha+\gamma)m+1-r;\nu x,\omega m x)}{\Gamma[\alpha+(\alpha+\gamma)m+1-r]}, (r = 1,\cdots,n) \qquad (3.4.15)$$

and

$$\Theta(x) = x^{\alpha-1}\sum_{m=0}^{\infty}\frac{\lambda^m x^{(\alpha+\gamma)m}\Phi_3(\beta\lambda,\alpha+(\alpha+\gamma)m;\nu x,\omega m x)}{\Gamma[\alpha+(\alpha+\gamma)m]} \qquad (3.4.16)$$

Proof 3.4.2. Projecting both sides of the integro-differential equation (3.4.12) to Laplace transform and using the known formulas

$$L\{D_t^\alpha f(t); s\} = s^\alpha F(s) - \sum_{r=1}^{n} s^{r-1}D_t^{\alpha-r}f(t)|_{t=0}, (r-1 < \alpha \le r; r \in \mathbb{N}) \qquad (3.4.17)$$

and (3.4.8), we find that

$$s^\alpha H(s) = \sum_{r=1}^{n} s^{r-1}D_\tau^{\alpha-r}h(\tau)|_{\tau=0} = \lambda s^{-\gamma}(1-\frac{\nu}{s})^{-\beta}\exp(\omega/s)H(s) + \mu F(s), \qquad (3.4.18)$$

Solving for $H(s)$, we obtain

$$H(s) = \sum_{r=1}^{n} b_r s^{r-1}[s^{\alpha} - \lambda s^{-\gamma}(1-\frac{\nu}{s})^{-\beta}e^{\frac{\omega}{s}}]^{-1} + \mu F(s)[s^{\alpha} - \lambda s^{-\gamma}(1-\frac{\nu}{s})^{-\beta}e^{\frac{\omega}{s}}]^{-1} \tag{3.4.19}$$

$$= \sum_{r=1}^{n} b_r \sum_{m=0}^{\infty} \lambda^m s^{-[\alpha+(\alpha+\gamma)m+1-r]}(1-\frac{\nu}{s})^{-m\beta}\exp(m\omega/s)$$
$$+ \mu F(s)\sum_{m=0}^{\infty} \lambda^m s^{-[\alpha+(\alpha+\gamma)m]}(1-\frac{\nu}{s})^{-\beta m}\exp[m\omega/s], \tag{3.4.20}$$

where we have tacitly assumed that

$$|\lambda s^{-\alpha-\gamma}(1-\frac{\nu}{s})^{-\beta}\exp(\omega/s)| < 1.$$

Taking inverse Laplace transform of both sides of (3.4.20) with the help of (3.4.8), the result (3.4.14) readily follows. In order to establish the uniqueness of the solution, we set $\zeta = \tau - t$ and apply the operator I_{τ}^{α} then after some calculations, the given equation (3.4.12) transforms into the form

$$h(\tau) = \sum_{r=1}^{n} b_r \frac{\tau^{\alpha-r}}{\Gamma(\alpha-r+1)}$$
$$+ \lambda\int_0^{\tau} h(\zeta)(t-\zeta)^{\gamma+\alpha-1}\Phi_3^*(\beta,\gamma+\alpha;\nu(\tau-\zeta),\omega(\tau-\zeta))d\zeta + \mu(I_{\tau}^{\alpha}f)(\tau). \tag{3.4.21}$$

where

$$\Phi_3^*(\beta,\gamma;\nu\tau,\omega\tau) = \frac{1}{\Gamma(\gamma)}\Phi_3(\beta,\gamma;\nu\tau,\omega\tau).$$

Since (3.4.21) is a Volterra integral equation with continuous kernel, it does admit a unique continuous solution, see Krasnov et al (1976).

Note 3.4.1. The solution of the Cauchy-type problem (3.4.12) and (3.4.13) can also be developed by the method of successive approximations, see for details , see (Kilbas et al, 2002) and Rall (1969).

Corollary 3.4.1. *Under the various relevant hypotheses of Theorem 3.4.1, there holds a unique continuous solution of the Cauchy-type problem associated with the Volterra-type integro-differential equation*

$$D_\tau^\alpha h(\tau) = \frac{\lambda}{\Gamma(\gamma)} \int_0^\tau t^{\gamma-1} \Phi(\beta, \gamma; \nu t) h(\tau - t) \mathrm{d}t + \mu f(\tau), (0 \le \tau \le 1), \tag{3.4.22}$$

together with the initial conditions (3.4.13), given by

$$h(\tau) = \sum_{r=1}^{n} b_r y_r(\tau) + \mu \int_0^\tau \Theta^*(\tau - t) f(t) \mathrm{d}t, \tag{3.4.23}$$

where

$$y_r(x) = \sum_{m=0}^{\infty} \frac{\lambda^m x^{(\alpha+\gamma)m+\alpha-r} \Phi(\beta m, \alpha + (\alpha+\gamma)m + 1 - r; \nu x)}{\Gamma[\alpha + (\alpha+\gamma)m + 1 - r]}, \tag{3.4.24}$$

and

$$\Theta^*(x) = x^{\alpha-1} \sum_{m=0}^{\infty} \frac{\lambda^m x^{(\alpha+\gamma)m} \Phi(\beta m, \alpha + (\alpha+\gamma)m; \nu x)}{\Gamma[\alpha + (\alpha+\gamma)m]}. \tag{3.4.25}$$

Corollary 3.4.2. *(Saxena et al, 2003, p.91). Under the various relevant hypotheses of Theorem 3.4.1, there holds a unique continuous solution of the Cauchy-type problem associated with the Volterra-type integro-differential equation*

$$D_\tau^\alpha h(\tau) = \frac{\lambda}{\Gamma(\gamma)} \int_0^\tau t^{\gamma-1} \Phi(\beta, \gamma; \nu t) h(\tau - t) \mathrm{d}t + \mu \frac{t^{\rho-1}}{\Gamma(\rho)} \phi(\sigma, \rho; \nu\tau), \tag{3.4.26}$$

$0 \le \tau \le 1$ *together with the initial conditions (3.4.13), given by*

$$h(\tau) = \sum_{r=1}^{n} b_r \Omega_r(\tau) + \mu \sum_{m=0}^{\infty} \frac{\lambda^m \tau^{\alpha(m+1)+mr+\rho-1} \Phi[\beta m + \sigma, (\alpha+\gamma)m + \alpha + \rho; \nu\tau]}{\Gamma[\alpha + (\alpha+\gamma)m + \rho]} \tag{3.4.27}$$

where

$$\Omega_r(x) = \sum_{m=0}^{\infty} \frac{\lambda^m x^{(\alpha+\gamma)m+\alpha-r} \Phi(\beta m, \alpha + (\alpha+\gamma)m + 1 - r; \nu x)}{\Gamma[\alpha + (\alpha+\gamma)m + 1 - r]}, (r = 1, \cdots, n). \tag{3.4.28}$$

Proof 3.4.3. If we set $f(x) = \frac{x^{\rho-1}}{\Gamma(\rho)}\Phi(\alpha,\rho;\nu x)$, and $\omega \to 0$, then by using the summation formula for the confluent hypergeometric function (Erdelyi, et al, 1953, p.271, eq.(5))

$$\int_0^t \frac{u^{c-1}}{\Gamma(c)}\Phi(a,c;u)\frac{(t-u)^{c'-1}}{\Gamma(c')}\Phi(a',c';t-u)\mathrm{d}u = \frac{t^{c+c'-1}}{\Gamma(c+c')}\Phi(a+a',c+c';t) \quad (3.4.29)$$

where $\Re(c) > 0, \Re(c') > 0$, we obtain Corollary 3.4.2.

Remark 3.4.2. If we take $\rho = 1$ in Corollary 3.4.2, it reduces to a result given by Al-Shammery et al (1999,p.503).

3.4.2. A Cauchy-type problem involving a Caputo derivative

Notation 3.4.8. ${}_0^cD_t^\alpha f(t), {}^cD_t^\alpha f(t), \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}f(t)$, Caputo derivative of $f(t)$ of order $\alpha > 0$.

Definition 3.4.8. According to Caputo (1969), the fractional derivative of order $\alpha > 0$ of a casual function $f(t)$, that is $f(t) = 0$ for $t < 0$, is defined as

$$ {}_0^cD_t^\alpha f(t) = ({}^cD_t^\alpha f)(t) = \frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^\tau \frac{f^{(m)}(u)}{(t-u)^{\alpha-m+1}}\mathrm{d}u, \quad (3.4.30)$$

$$(m-1 < \alpha < m, m \in N),$$

where $f^{(m)}(t)$ denotes the usual derivative of $f(t)$ of order $m \in N_0$.

By virtue of (3.4.11) and (3.4.30), we obtain the Laplace transform of Caputo derivative in the form

$$L\left\{\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}f(t);s\right\} = s^\alpha F(s) - \sum_{r=0}^{m-1} s^{\alpha-r-1}f^{(r)}(0); m-1 < \alpha \le m; m \in N, \quad (3.4.31)$$

which is more suited for initial-value problems than (3.4.17), and where $F(s)$ is the Laplace transform of $f(t)$.

Theorem 3.4.2. *Let $\beta,\gamma,\lambda,\mu \in C, \nu,\omega \in R, \Re(\alpha) > 0, \Re(\gamma) > 0$, and $f(t)$ is assumed to be continuous on every finite interval $[0,T], 0 < T < \infty$, and of the exponential order $\mathrm{e}^{\eta\tau}$, when $\tau \to \infty$. Then for the Cauchy-type problem for the fractional integro-differential equation of Volterra type*

$$\frac{d^\alpha}{d\tau^\alpha}h(\tau) = \frac{\lambda}{\Gamma(\gamma)}\int_0^\tau t^{\gamma-1}\Phi_3(\beta,\gamma;\nu t,\omega t)h(\tau-t)dt + \mu f(\tau), \tag{3.4.32}$$

where $0 \le \tau \le 1$*, together with the initial conditions*

$$\frac{d^r}{d\tau^r}h(\tau)|_{\tau=0} = a_r, r = 0,\cdots,n-1; (n-1 < \alpha \le n); n \in N \tag{3.4.33}$$

$a_0,\cdots,a_{n-1} \in \mathcal{R}$*, there exists a unique continuous solution, given by*

$$h(\tau) = \sum_{r=0}^{n-1} a_r y_r^*(\tau) + \mu\int_0^\tau \Theta(\tau-t)f(t)dt, \tag{3.4.34}$$

where

$$y_r^*(x) = \sum_{m=0}^{\infty}\frac{\lambda^m x^{(\alpha+\gamma)m+r}\Phi_3(\beta m,(\alpha+\gamma)m+r+1;\nu x,\omega m x)}{\Gamma[(\alpha+\gamma)m+r+1]} \tag{3.4.35}$$

and $\Theta(x)$ *is defined in (3.4.16).*

Exercises 3.4.

3.4.1. Under the various hypotheses of Theorem 3.4.1, the Cauchy-type problem for the fractional integro-differential equation of Volterra type

$$D_\tau^\alpha h(\tau) = \frac{\lambda}{\Gamma(\gamma)}\int_0^\tau t^{\gamma-1}\exp(\nu t)h(\tau-t)dt + \mu f(\tau), (0 \le \tau \le 1), \tag{3.4.36}$$

together with the initial conditions (3.4.12), has a unique continuous solution given by

$$h(\tau) = \sum_{r=1}^{n} b_r \Im_r(\tau) + \mu\int_0^\tau \aleph(\tau-t)f(t)dt, \tag{3.4.37}$$

where

$$\Im_r(x) = \sum_{m=0}^{\infty}\frac{\lambda^m x^{(\alpha+\gamma)m+\alpha-r}\Phi(\gamma m,\alpha+(\alpha+\gamma)m+1-r;\nu x)}{\Gamma[\alpha+(\alpha+\gamma)m+1-r]}, (r = 1,\cdots,n) \tag{3.4.38}$$

and

$$\aleph(x) = \sum_{m=0}^{\infty}\frac{\lambda^m x^{(\alpha+\gamma)m+\alpha-1}\Phi(\gamma m,\alpha+(\alpha+\gamma)m;\nu x)}{\Gamma[\alpha+(\alpha+\gamma)m]} \tag{3.4.39}$$

Hence or otherwise deduce the special case of the above problem for $f(\tau) = \exp(\nu\tau)$, given by Al-Shammery et al (2000, p.82), which itself is a generalization of a result given by Boyadjiev et al (1997, p.4).

3.4.2. Under the various hypotheses of Theorem 3.4.2, the Cauchy-type problem for the fractional integro-differential equation of Volterra type

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}\tau^\alpha}h(\tau) = \frac{\lambda}{\Gamma(\gamma)}\int_0^\tau t^{\gamma-1}\Phi(\beta,\gamma;\nu t)h(\tau-t)\mathrm{d}t + \mu f(\tau), (0 \le \tau \le 1) \tag{3.4.40}$$

together with the initial conditions

$$\frac{\mathrm{d}^r}{\mathrm{d}\tau^r}h(\tau)|_{\tau=0} = a_r, r = 0,\cdots,n-1; (n-1 < \alpha \le n); n \in N \tag{3.4.41}$$

where $a_-,\cdots,a_{n-1} \in \mathcal{R}$ has a unique continuous solution given by

$$h(\tau) = \sum_{r=0}^{n-1} a_r y_r^*(\tau) + \mu\int_0^\tau \Theta^*(\tau-t)f(t)\mathrm{d}t, \tag{3.4.42}$$

where

$$y_r^*(x) = \sum_{m=0}^{\infty} \frac{\lambda^r x^{(\alpha+\gamma)m+r}\Phi(\beta r, (\alpha+\gamma)m + r + 1; \nu x)}{\Gamma[(\alpha+\gamma)m + r + 1]}, (r = 0,\cdots,n-1) \tag{3.4.43}$$

and $\Theta^*(x)$ is defined in (3.4.25).

3.5. Fractional Kinetic and Fractional Diffusion Equations

Fractional kinetic equations have been studied to describe certain physical phenomena such as diffusion in porous media with fractal geometry, kinematics in viscoelastic media, relaxation processes in viscoelastic materials, glassy materials, synthetic polymers etc. We now proceed to prove a general theorem on fractional kinetic equation.

Notation 3.5.1. $R_{\nu,\mu}(a,c,t)$ Lorenzo and Hartley function

Definition 3.5.1.

$$R_{\nu,\mu}(a,c,t) = \sum_{n=0}^{\infty} \frac{a^n (t-c)^{(n+1)\nu-\varpi-1}}{\Gamma[(n+1)\nu-\mu]}$$

Theorem 3.5.1. *Let $\Re(\nu) > 0, c > 0$ and let $f(x) \in \mathcal{R}_+$, then the fractional kinetic equation (Hille and Tamarkin,1930)*

$$N(t) - N_0 f(t) = -c^{\nu}{}_0D_t^{-\nu}N(t), \tag{3.5.1}$$

is solvable, and there exists a solution given by the formula

$$N(t) = N_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_0^{\tau} f(u) E_{\nu}[-c^{\nu}(t-u)^{\nu}]\mathrm{d}u, \tag{3.5.2}$$

Proof 3.5.1. Applying Laplace transform to both sides of (3.5.1), we obtain

$$N^{\sim}(s) - N_0 f^{\sim}(s) = -c^{\nu}s^{-\nu}N^{\sim}(s), \tag{3.5.3}$$

where $f^{\sim}(s)$ denotes the Laplace transform of $f(t)$.

Solving for $N^{\sim}(s)$ and taking its inverse Laplace transform, we arrive at the desired result (3.5.2).

Remark 3.5.1. The result (3.5.2) has been obtained by Saxena et al (2004) in a different form involving the H-function, see Exercise 3.5.2.

Theorem 3.5.2. *If $c > 0, b \geq 0, \Re(s) > 0, \nu > \mu + 1$, then for the solution of*

$$N(t) - N_0 R_{\nu,\mu}(-c^{\nu}, b, t) = -c^{\nu}{}_0D_t^{-\nu}N(t), \tag{3.5.4}$$

there holds the formula (Saxena et al , 2004a)

$$N(t) = \frac{N_0}{\nu}(t-b)^{\nu-\mu-1}[E_{\nu,\nu-\mu-1}\{-c^{\nu}(t-b)^{\nu} + (1+\mu)\}E_{\nu,\nu-\mu}(-c^{\nu}(t-b)^{\nu})]. \tag{3.5.5}$$

Hint: The Laplace transform of $R_{\nu,\mu}(a,c,t)$ is given by (Lorenzo and Hartley, 1999)

$$L\{R_{\nu,\mu}(a,0,t);s\} = \frac{\mathrm{e}^{-cs}s^{\mu}}{s^{\nu}-a}, \quad \Re(\nu-\mu) > 0, \Re(s) > 0. \tag{3.5.6}$$

Exercises 3.5.

3.5.1. Show that the solution of the following initial value problem for the fractional diffusion equation in one dimension (Schneider and Wyss,1989)

$$N(x,t) = \varphi(x) + \lambda^2 {}_0D_t^{-\alpha} \frac{\partial^2 N(x,t)}{\partial x^2}, (t > 0, -\infty < x < \infty), \tag{3.5.7}$$

with initial conditions

$$\lim_{x\to\pm\infty} N(x,t) = 0, N(x,0) = \varphi(x),$$

is given by

$$N(x,t) = \int_{-\infty}^{\infty} G(x-\xi)\varphi(\xi)\mathrm{d}\xi, \tag{3.5.8}$$

where

$$G(x,t) = \frac{1}{\pi}\int_0^{\infty} E_{\alpha,1}(-\lambda^2 k^2 t^\alpha)\cos(kx)\mathrm{d}k. \tag{3.5.9}$$

3.5.2. Let $\Re(\nu) > 0, c > 0$ and let $f(x) \in \mathcal{R}_+$, then the fractional kinetic equation associated with Riemann-Liouville fractional integral

$$N(t) - N_0 f(t) = -c^\nu {}_0D_t^{-\nu} N(t), \tag{3.5.10}$$

is solvable, and there exists a solution given by the formula (Saxena, et al,2004b)

$$N(t) = N_0 \int_0^{\tau} H_{1,2}^{1,1}\left[c^\nu (t-\tau)^\nu \Big|_{(-1/\nu,1),(0,\nu)}^{(-1/\nu,1)}\right] f(\tau)\mathrm{d}\tau. \tag{3.5.11}$$

Derive the solutions in the following cases: (i) $f(t) = 1$, (ii) $f(t) = t^{\rho-1}$.

3.5.3. Solve Theorem 3.5.1 completely.

3.5.4. Prove that the solution to the Cauchy-type problem associated with Riemann-Liouville fractional derivative

$$(D_{0+}^{\alpha} f)(x) - \omega f(x) = g(x), (D_{0+}^{\alpha-1} f)(0+) = b, \tag{3.5.12}$$

with $0 < \alpha < 1$ and $\omega, b \in R$ is given by

$$f(x) = bx^{\alpha-1}E_{\alpha,\alpha}[\omega x^\alpha] + \int_0^x (x-t)^{\alpha-1}E_{\alpha,\alpha}[\omega(x-t)^\alpha]g(t)\mathrm{d}t. \tag{3.5.13}$$

Also derive the solution, when $g(x) = 0$.

3.5.5. Prove that the solution to the Cauchy-type problem associated with Riemann-Liouville fractional derivative

$$(D_{0+}^{\alpha} f)(x) - \omega f(x) = g(x), (D_{0+}^{\alpha-1} f)(0+) = b, (D_{0+}^{\alpha-2} f)(0+) = c, \tag{3.5.14}$$

with $1 < \alpha < 2$ and $\omega, b, c \in R$ is given by

$$f(x) = bx^{\alpha-1}E_{\alpha,\alpha}(\omega x^{\alpha}) + cx^{\alpha-2}E_{\alpha,\alpha-1}(\omega x^{\alpha}) + \int_0^x (x-t)^{\alpha-1}E_{\alpha,\alpha}[\omega(x-t)^{\alpha}]g(t)\mathrm{d}t. \tag{3.5.15}$$

Also derive the solution, when $g(x) = 0$.

3.5.6. Prove that the solution to the Cauchy-type problem associated with Caputo fractional derivative

$$({}^{c}D_{0+}^{\alpha} f)(x) - \omega f(x) = g(x), f(0) = b, \tag{3.5.16}$$

with $0 < \alpha < 1$ and $\omega, b \in R$ is given by

$$f(x) = bE_{\alpha}[\omega x^{\alpha}] + \int_0^x (x-t)^{\alpha-1}E_{\alpha,\alpha}[\omega(x-t)^{\alpha}]g(t)\mathrm{d}t. \tag{3.5.17}$$

Also derive the solution when $g(x) = 0$.

3.5.7. Prove that the solution to the Cauchy-type problem associated with Caputo fractional derivative

$$({}^{c}D_{0+}^{\alpha} f)(x) - \omega f(x) = g(x); f(0) = b, f'(0) = c, \tag{3.5.18}$$

with $1 < \alpha < 2$ and $\omega, b, c \in R$ is given by

$$f(x) = bE_{\alpha}(\omega x^{\alpha}) + cxE_{\alpha,2}(\omega x^{\alpha}) + \int_0^x (x-t)^{\alpha-1}E_{\alpha,\alpha}[\omega(x-t)^{\alpha}]g(t)\mathrm{d}t. \tag{3.5.19}$$

Also derive the solution, when $g(x) = 0$.

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