CHAPTER 5

BASIC HYPERGEOMETEIC FUNCTIONS

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5.0. Introduction

C.F. Gauss (1812) introduced the following series, known as Gaussian hypergeometric series (function)

$$\begin{aligned} 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots \\ = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\frac{z^n}{n!} \\ = {}_2F_1(a,b;c;z) \equiv {}_2F_1\left(\begin{matrix} a, b; z \\ c \end{matrix}\right) \end{aligned} \tag{5.0.1}$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1), n > 0, (a)_0 = 1.$

Ratio test will reveal that (5.0.1) is convergent for $|z| < 1$ and for $z = 1$ if $\Re(c-a-b) > 0$ and for $z = -1$ if $\Re(c-a-b+1) > 0$.

This case, further, is extended to

$${}_pF_q(a_1, a_2, \cdots, a_p; b_1, b_2, \cdots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n}\frac{z^n}{n!}. \tag{5.0.2}$$

The series (5.0.2) is convergent for all values of z when $p < q+1$. Also, it is valid for $|z| < 1$ when $p = q+1$ and for $\Re\left(\sum_{r=1}^{q} b_r - \sum_{s=1}^{p} a_s\right) > 0$ when $z = 1$.

Heine (1898) generalized the Gaussian hypergeometric function with the help of basic number, say, $[\alpha]_q$ which he introduces as

$$[\alpha]_q = \frac{1-q^\alpha}{1-q}, \quad |q| < 1 \tag{5.0.3}$$

and α any number, real or complex. It is evident that, as $q \to 1$, the basic number

$$[\alpha]_q \to \alpha.$$

Applying the concept of basic number, Heine introduced the q-Gaussian function or q-(or basic) hypergeometric function by defining the series,

$$1 + \frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)}z + \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})}z^2 + \cdots \tag{5.0.4}$$

with $|q| < 1$, called the base of the basic hypergeometric functions. As $q \to 1$, (5.0.4) reduces to (5.0.1).

Such series are found in the work of Euler (1748), Gauss (1866). Jacobi (1829) in his “Fundamenta Nova” defined four theta functions with the help of basic hypergeometric series.

5.0.1. Convergence of Heine series

We apply comparison test to study the convergence of (5.0.4). Thus, if we denote by u_n the n^{th} term of (5.0.4), we get

$$\frac{U_{n+1}}{U_n} = \frac{(1-q^{a+n})(1-q^{b+n})}{(1-q^n)(1-q^{c+n})}z. \tag{5.0.5}$$

Now, with $|q| < 1$, as $n \to \infty$, we get

$$\frac{U_{n+1}}{U_n} \to z.$$

Hence the series (5.0.4) converges absolutely for $|z| < 1$.

Normally, we follow the notation of G.N. Watson who replaced q^a by a. As such Heine's series can be put in the form,

$$\begin{aligned}1 &+ \frac{(1-a)(1-b)}{(1-q)(1-c)}z + \frac{(1-a)(1-aq)(1-b)(1-bq)}{(1-q)(1-q^2)(1-c)(1-cq)}z^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{[a;q]_n[b;q]_n}{[q;q]_n[c;q]_n} z^n \\ &= {}_2\phi_1[a,b;c;q;z] = {}_2\phi_1\begin{bmatrix} a, & b; & q; & z \\ & c & & \end{bmatrix} \qquad (5.0.6)\end{aligned}$$

where

$$[a;q]_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), n \geq 1$$

and

$$[a;q]_0 = 1.$$

Following the above notation, we can define a generalized basic hypergeometric functions (series) as

$$\begin{aligned}{}_r\phi_s\begin{bmatrix} a_1, & a_2, & \cdots, & a_r; & q;z \\ b_1, & b_2, & \cdots, & b_s & \end{bmatrix} &= {}_r\phi_s\begin{bmatrix} (a_r); & q; & z \\ (b_r) & & \end{bmatrix} \\ &= \sum_{n=0}^{\infty} \frac{[a_1;q]_n \cdots [a_r;q]_n}{[q;q]_n[b_1;q]_n \cdots [b_s;q]_n} z^n. \qquad (5.0.7)\end{aligned}$$

If there is no ambiguity, we drop q from the $[\alpha, q]_n$ and simply write it as $[\alpha]_n$ and drop q from the ϕ notation also.

5.1. Certain Summations

In this section we can discuss certain simple but very important summation formulae for basic hypergeometric functions. Let us consider the following functions,

$$ {}_1\phi_0\left[a;\ ;q;z\right] = \sum_{n=0}^{\infty} \frac{[a]_n}{[q]_n} z^n \tag{5.1.1} $$

$$ {}_1\phi_0\left[a;\ ;q;qz\right] = \sum_{n=0}^{\infty} \frac{[a]_n}{[q]_n} z^n q^n \tag{5.1.2} $$

Now, (5.1.1) and (5.1.2) yield,

$$ \begin{aligned} {}_1\phi_0\left[a;\ ;q;z\right] - {}_1\phi_0\left[a;\ ;q;qz\right] &= \sum_{n=0}^{\infty} \frac{[a]_n}{[q]_n} z^n (1-q^n) = \sum_{n=1}^{\infty} \frac{[a]_n}{[q]_{n-1}} z^n \\ &= (1-a)z \sum_{n=0}^{\infty} \frac{[aq]_n}{[q]_n} z^n \\ &= (1-a)z \ \ {}_1\phi_0\left[aq;\ ;q;z\right]. \end{aligned} $$

Hence, we get,

$$ {}_1\phi_0\left[a;\ ;q;z\right] - {}_1\phi_0\left[a;\ ;q;qz\right] = (1-a)z \ \ {}_1\phi_0\left[aq;\ ;q;qz\right]. \tag{5.1.3} $$

Similarly, we have

$$ \begin{aligned} {}_1\phi_0\left[a;\ ;q;z\right] - a \ \ {}_1\phi_0\left[a;\ ;q;qz\right] &= \sum_{n=0}^{\infty} \frac{[a]_n z^n}{[q]_n}(1-aq^n) = \sum_{n=0}^{\infty} \frac{[a]_{n+1} z^n}{[q]_n} \\ &= (1-a) \sum_{n=0}^{\infty} \frac{[aq]_n z^n}{[q]_n} = (1-a) \ \ {}_1\phi_0\left[aq;\ ;q;z\right]. \end{aligned} $$

Hence, we have

$$ {}_1\phi_0\left[a;\ ;q;z\right] - a \ \ {}_1\phi_0\left[a;\ ;q;qz\right] = (1-a) \ \ {}_1\phi_0\left[aq;\ ;q;z\right]. \tag{5.1.4} $$

Now, eliminating ${}_1\phi_0\left[aq;\ ;q;z\right]$ with the help of (5.1.3) and (5.1.4), we get,

$$ {}_1\phi_0\left[a;\ ;q;z\right] - {}_1\phi_0\left[a;\ ;q;qz\right] = z \ \ {}_1\phi_0\left[a;\ ;q;z\right] - az \ \ {}_1\phi_0\left[a;\ ;q;qz\right] $$

which leads to,

$$(1-z)\ {}_1\phi_0[a;\ ;q;z] = (1-az)\ {}_1\phi_0[a;\ ;q;qz] \tag{5.1.5}$$

Hence, we have,

$${}_1\phi_0[a;\ ;q;z] = \frac{(1-az)}{(1-z)}\ {}_1\phi_0[a;\ ;q;qz]. \tag{5.1.6}$$

Now, replacing z by zq in (5.1.6), we get,

$${}_1\phi_0[a;\ ;q;zq] = \frac{(1-azq)}{(1-zq)}\ {}_1\phi_0[a;\ ;q;q^2z]. \tag{5.1.7}$$

Substituting for ${}_1\phi_0[a;\ ;q;zq]$ from (5.1.7) on the right hand side of (5.1.6) we get

$${}_1\phi_0[a;\ ;q;z] = \frac{(1-az)(1-azq)}{(1-z)(1-zq)}\ {}_1\phi_0[a;\ ;q;q^2z]. \tag{5.1.8}$$

Now, by iteration, we get

$${}_1\phi_0[a;\ ;q;z] = \frac{(1-az)(1-azq)\cdots(1-azq^{n-1})}{(1-z)(1-zq)\cdots(1-zq^{n-1})}{}_1\phi_0[a;\ ;q;q^nz]. \tag{5.1.9}$$

Now, letting $n \to \infty$ in (5.1.9) the ${}_1\phi_0$ reduces to 1 and we get,

$${}_1\phi_0[a;\ ;q;z] = \frac{[az;q]_\infty}{[z]_\infty} \tag{5.1.10}$$

where

$$[\alpha;q]_\infty = \prod_{r=0}^{\infty}(1-\alpha q^r).$$

The summation given by (5.1.10) is q-analogue of binomial theorem.

Exercises 5.1.

Prove the following:

5.1.1.

$${}_1\phi_0[a;\ ;q;z]\ {}_1\phi_0[b;\ ;q;az] = {}_1\phi_0[ab;\ ;q;z]. \tag{5.1.11}$$

5.1.2.

$$e_q(z) = {}_1\phi_0[\ ;\ ; q; z] = \frac{1}{[z;q]_\infty} \tag{5.1.12}$$

(q- analogue of exponential). (Hint: put $a = 0$ in (5.1.10)).

5.1.3.

$$1 + \frac{z}{1-q} + \frac{z^2}{(1-q)(1-q^2)} + \cdots + \frac{z^n}{(1-q)(1-q^2)\cdots(1-q^n)} + \cdots \tag{5.1.13}$$
$$= \frac{1}{(1-z)(1-zq)(1-zq^2)(1-zq^3)\cdots(1-zq^n)\cdots}.$$

(It is (5.1.12) in series and product forms).

5.1.4.

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}(-z)^n}{[q]_n} = [z;q]_\infty \tag{5.1.14}$$

(big exponential). (Hint: Replace z by z/a in (5.1.10) and let $a \to \infty$.)

5.1.5.

$$1 + \frac{q}{1-q} + \frac{q^2}{(1-q)(1-q^2)} + \cdots = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)} \tag{5.1.15}$$

(Hint: put $z = q$ in (5.1.12)).

5.1.6.

$$1 - \frac{z}{1-q} + \frac{qz^2}{(1-q)(1-q^2)} - \cdots + \frac{(-1)^n q^{n(n-1)/2} z^n}{(1-q)(1-q^2)\cdots(1-q^n)} = (1-z)(1-zq)(1-zq^2)\cdots . \tag{5.1.16}$$

(Hint: Expand (5.1.14)).

5.1.7.

$$1 - \frac{q}{1-q} + \frac{q^2}{(1-q)(1-q^2)} - \cdots + \frac{(-1)^n q^{n(n+1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)} + \cdots = (1-q)(1-q^2)(1-q^3). \tag{5.1.17}$$

5.1.1. Summation of ${}_2\phi_1[a, b; c; z]$

In this section we shall discuss the summation of ${}_2\phi_1[a, b; c; z]$. We have

$$\begin{aligned}
{}_2\phi_1[a, b; c; z] &= \sum_{n=0}^{\infty} \frac{[a]_n[b]_n z^n}{[q]_n[c]_n} \\
&= \sum \frac{[a]_n z^n}{[q]_n} \frac{[b]_n[bq^n]_\infty}{[c]_n[cq^n]_\infty} \frac{[cq^n]_\infty}{[bq^n]_\infty} \\
&= \frac{[b]_\infty}{[c]_\infty} \sum \frac{[a]_n z^n}{[q]_n} \frac{[cq^n]_\infty}{[bq^n]_\infty} \\
&= \frac{[b]_\infty}{[c]_\infty} \sum_{n=0}^{\infty} \frac{[a]_n z^n}{[q]_n} \sum_{r=0}^{\infty} \frac{[c/b]_r b^r q^{nr}}{[q]_r} \quad \text{(using (5.1.10))} \\
&= \frac{[b]_\infty}{[c]_\infty} \sum_{r=0}^{\infty} \frac{[c/b]_r b^r}{[q]_r} \sum_{n=0}^{\infty} \frac{[a]_n (zq^r)^n}{[q]_n} \\
&= \frac{[b]_\infty}{[c]_\infty} \sum_{r=0}^{\infty} \frac{[c/b]_r b^r}{[q]_r} \frac{[azq^r]_\infty}{[zq^r]_\infty} \quad \text{(using (5.1.10))} \\
&= \frac{[b]_\infty}{[c]_\infty} \frac{[az]_\infty}{[z]_\infty} \sum_{r=0}^{\infty} \frac{[c/b]_r[z]_r b^r}{[q]_r[az]_r}
\end{aligned}$$

or

$${}_2\phi_1\begin{bmatrix} a, & b; & z \\ & c & \end{bmatrix} = \frac{[b]_\infty[az]_\infty}{[c]_\infty[z]_\infty} {}_2\phi_1\begin{bmatrix} c/b, & z; & b \\ & az & \end{bmatrix}. \tag{5.1.18}$$

The right side reduces to ${}_1\phi_0$ for $z = c/ab$ and can be summed with the help of (5.1.10) to yield

$${}_2\phi_1\begin{bmatrix} a, & b; & c/ab \\ & c & \end{bmatrix} = \frac{[c/a]_\infty[c/b]_\infty}{[c]_\infty[c/ab]_\infty}. \tag{5.1.19}$$

This is the required sum, the basic analogue of Gauss' theorem. As $q \to 1$, (5.1.19) reduces to the Gaussian sum

$${}_2F_1\begin{bmatrix} a, & b; & 1 \\ & c & \end{bmatrix} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{5.1.20}$$

valid for $\Re(c - a - b) > 0$.

Again, setting $c = abz$ in (5.1.19) and then letting $b \to 0$, we get (5.1.10).

Exercises 5.1.

5.1.8. Show that

$$ {}_1\phi_1\begin{bmatrix} a; & -c/a \\ c; & q \end{bmatrix} = \frac{[c/a]_\infty}{[c]_\infty}. \tag{5.1.21}$$

(A series

$$ {}_r\phi_s\begin{bmatrix} (a_r); & q; & z \\ (b_s); & q^\lambda & \end{bmatrix} = \sum_{n=0}^{\infty} \frac{[(a_r)]_n z^n q^{\lambda n(n-1)/2}}{[q]_n[(b_s)]_n}$$

is called an abnormal q-series for $\lambda > 0$ and is rapidly convergent because of $|q| < 1$.)

5.1.9. Show that

$$ {}_0\phi_1\begin{bmatrix} ; & c \\ c; & q^2 \end{bmatrix} = \sum_{n=0}^{\infty} \frac{c^n q^{n^2-n}}{[q]_n[c]_n} = \frac{1}{[c]_\infty}. \tag{5.1.22}$$

5.1.10. Show that

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{[q]_n^2} = \frac{1}{[q]_\infty}. \tag{5.1.23}$$

5.1.11. Show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{[q^2;q^2]_n} = \frac{1}{[-q;q]_n} = [q;q^2]_\infty. \tag{5.1.24}$$

5.2. Continued Fractions and Basic Hypergeometric Functions

We define a finite continued fraction as

$$a_0 + \frac{a_1}{b_1 +}\;\frac{a_2}{b_2 +}\;\frac{a_3}{b_3 +}\cdots\frac{a_n}{+b_n}. \tag{5.2.1}$$

It will be called an infinite continued fraction when $n \to \infty$. Suppose we want to express $\sqrt{2}$ in terms of continued fraction, then we have,

$$\sqrt{2} = 1 + \sqrt{2} - 1 = 1 + (\sqrt{2} - 1)\frac{(\sqrt{2}+1)}{\sqrt{2}+1} = \frac{1}{1+\sqrt{2}}$$

$$= 1 + \cfrac{1}{1 + 1 + \cfrac{1}{1+\sqrt{2}}} = 1 + \cfrac{1}{2 + \cfrac{1}{1+\sqrt{2}}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}} \tag{5.2.2}$$

It provides fairly a large degree of approximation. Before we go further, we establish certain results to be used in our analysis. We start with (5.1.18). That is,

$${}_2\phi_1\begin{bmatrix} a, & b; & z \\ & c & \end{bmatrix} = \frac{[b]_\infty[az]_\infty}{[c]_\infty[z]_\infty}\,{}_2\phi_1\begin{bmatrix} c/b, & z; & b \\ & az & \end{bmatrix}. \tag{5.2.3}$$

Now, applying this transformation on the right side we get,

$${}_2\phi_1\begin{bmatrix} a, & b; & z \\ & c & \end{bmatrix} = \frac{[c/b]_\infty[bz]_\infty}{[c]_\infty[z]_\infty}\,{}_2\phi_1\begin{bmatrix} abz/c, & b; & c/b \\ & bz & \end{bmatrix}. \tag{5.2.4}$$

(This is known as Euler's transformation). Hypergeometric functions and continued fractions have very intimate relations. We shall illustrate it with the help of a general example which generalizes scores of results of S. Ramanujan.

Let, for i a non-negative integer

$$F_i = \sum_{n=0}^{\infty} \frac{[\alpha q^i]_n[\beta q^i]_n x^n}{[q]_n[\gamma]_{n+i}} \tag{5.2.5}$$

$$= \frac{1}{[\gamma]_i}\,{}_2\phi_1\begin{bmatrix} \alpha q^i, & \beta q^i; & x \\ & \gamma q^i & \end{bmatrix} \tag{5.2.6}$$

and

$$H_i = \sum_{n=0}^{\infty} \frac{[\alpha q^i]_n [\beta q^i]_n}{[q]_n [\gamma]_{n+i}} (xq)^n \tag{5.2.7}$$

$$= \frac{1}{[\gamma]_i} {}_2\phi_1 \begin{bmatrix} \alpha q^i, & \beta q^i; & xq \\ & \gamma q^i & \end{bmatrix}. \tag{5.2.8}$$

Now,

$$\begin{aligned} F_i - H_i = &= \sum_{n=0}^{\infty} \frac{[\alpha q^i]_n [\beta q^i]_n}{[q]_n [\gamma]_{n+i}} (1 - q^n) x^n \\ &= \sum_{n=0}^{\infty} \frac{[\alpha q^i]_{n+1} [\beta q^i]_{n+1} x^{n+1}}{[q]_n [\gamma]_{n+i+1}} \\ &= (1 - \alpha q^i)(1 - \beta q^i) x \sum_{n=0}^{\infty} \frac{[\alpha q^{i+1}]_n [\beta q^{i+1}]_n x^n}{[q]_n [\gamma]_{n+i+1}} \end{aligned}$$

or

$$F_i - H_i = = (1 - \alpha q^i)(1 - \beta q^i) x F_{i+1}. \tag{5.2.9}$$

Now, we use (5.2.4) to transform F_i and H_i to get,

$$F_i = \frac{[\gamma/\beta]_\infty [x\beta q^i]_\infty}{[\gamma]_\infty [x]_\infty} \sum_{n=0}^{\infty} \frac{[\beta q^i]_n [\alpha\beta x q^i/\gamma]_n (\frac{\gamma}{\beta})^n}{[q]_n [\beta x q^i]_n} \tag{5.2.10}$$

and

$$H_i = \frac{[\gamma/\beta]_\infty [x\beta q^{i+1}]_\infty}{[\gamma]_\infty [xq]_\infty} \sum_{n=0}^{\infty} \frac{[\beta q^i]_n [\alpha\beta x q^{i+1}]_n (\frac{\gamma}{\beta})^n}{[q]_n [\beta x q^{i+1}]_n}. \tag{5.2.11}$$

Now (5.2.10) and (5.2.11) lead to

$$H_i - (1-x)F_{i+1} = \frac{[\gamma/\beta]_\infty [x\beta q^{i+1}]_\infty (-\beta q^i)}{[\gamma]_\infty [xq]_\infty} \sum_{n=0}^{\infty} \frac{[\beta q^{i+1}]_{n-1} [\alpha\beta x q^{i+1}/\gamma]_n (\frac{\gamma}{\beta})^n (1-q^n)}{[q]_n [\beta x q^{i+1}]_n}$$
$$= \frac{[\gamma/\beta]_\infty [x\beta q^{i+1}]_\infty (\gamma q^i)}{[\gamma]_\infty [xq]_\infty} \sum_{n=0}^{\infty} \frac{[\beta q^{i+1}]_n [\alpha\beta x q^{i+1}/\gamma]_{n+1} (\frac{\gamma}{\beta})^n}{[q]_n [\beta x q^{i+1}]_{n+1}}$$

or

$$H_i - (1-x)F_{i+1} = (\alpha\beta x q^{2i+1} - \gamma q^i) H_{i+1}. \tag{5.2.12}$$

Now, (5.2.9) and (5.2.12) respectively yield

$$\frac{F_i}{H_i} = 1 + \frac{(1-\alpha q^i)(1-\beta q^i)x}{H_i/F_{i+1}} \tag{5.2.13}$$

and

$$\frac{H_i}{F_{i+1}} = 1 - x + \frac{(\alpha\beta x q^{2i+1} - \gamma q^i)}{F_{i+1}/H_{i+1}}. \tag{5.2.14}$$

Now, iterating (5.2.13) and (5.2.14), we get,

$$\frac{F_i}{H_i} = 1 + \frac{(1-\alpha q^i)(1-\beta q^i)x}{1-x \quad +} \; \frac{\alpha\beta x q^{2i+1} - \gamma q^i}{1 \quad +} \; \frac{(1-\alpha q^{i+1})(1-\beta q^{i+1})x}{1-x \quad +} \; \frac{\alpha\beta x q^{2i+3} - \gamma q^{i+1}}{1 \quad +} \; \frac{(1-\alpha q^{i+2})(1-\beta q^{i+2})x}{1-x \quad +} \; \frac{\alpha\beta x q^{2i+5} - \gamma q^{i+2}}{1 \; + \; \cdots}. \tag{5.2.15}$$

This provides an infinite family of continued fraction representation for the ratio F_i/H_i. Taking $i = 0$, we get

$$ {}_2\phi_1 \begin{bmatrix} \alpha, & \beta; & x \\ & \gamma & \end{bmatrix} \Big/ {}_2\phi_1 \begin{bmatrix} \alpha, & \beta; & xq \\ & \gamma & \end{bmatrix}$$
$$= 1 + \frac{(1-\alpha)(1-\beta)x}{1-x \quad +} \; \frac{\alpha\beta x q - \gamma}{1 \quad +} \; \frac{(1-\alpha q)(1-\beta q)x}{1-x \quad +} \; \frac{\alpha\beta x q^3 - \gamma q}{1 \quad +} \; \frac{(1-\alpha q^2)(1-\beta q^2)x}{1-x \quad +} \; \frac{\alpha\beta x q^5 - \gamma q^2}{1 + \cdots}. \tag{5.2.16}$$

(5.2.16) includes scores of results due to S. Ramanujan as its special cases.

Exercises 5.2.

5.2.1. Show that

$$\sum_{n=0}^{\infty}\frac{q^{n^2}x^n}{[q]_n[\gamma_n]}\Big/\sum_{n=0}^{\infty}\frac{q^{n^2-n}x^n}{[q]_n[\gamma_n]}$$
$$=\frac{1}{1+}\ \frac{x}{1+}\ \frac{\gamma q-\gamma}{1\quad +}\ \frac{xq^2}{1+}\ \frac{xq^3-\gamma q}{1\quad +}\ \frac{xq^4}{1+\cdots}. \qquad (5.2.17)$$

(Putting $\gamma = -bq$ yields a famous result due to Ramanujan).

5.2.2. Show that

$$\frac{[xq;q^2]_\infty}{[-x;q^2]_\infty}=\sum_{n=0}^{\infty}\frac{[q;q^2]_n(-x)^n}{[q^2;q^2]_n}$$
$$=\frac{1}{1+}\ \frac{x}{1+}\ \frac{xq+q}{1\ +}\ \frac{xq^2}{1+}\ \frac{xq^3+q^2}{1\quad +}\ \frac{xq^4}{1+}\ \frac{xq^5+q^3}{1+\cdots}. \qquad (5.2.18)$$

5.2.3. Show that

$$\frac{[q^3;q^4]_\infty}{[-q;q^4]_\infty}=\frac{1}{1+}\ \frac{q}{1+}\ \frac{q^2+q^3}{1\quad +}\ \frac{q^5}{1+}\ \frac{q^4+q^7}{1\ +\cdots}. \qquad (5.2.19)$$

5.2.4. Show that

$$\sum_{n=0}^{\infty}\frac{x^nq^{n^2+n}}{[q]_n}\Big/\sum_{n=0}^{\infty}\frac{x^nq^{n^2}}{[q]_n}=\frac{1}{1+}\ \frac{xq}{1+}\ \frac{xq^2}{1+}\ \frac{xq^3}{1+}\ \frac{xq^4}{1\ +\cdots}. \qquad (5.2.20)$$

5.2.5. Show that

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{[q]_n}\Big/\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{[q]_n}=\frac{[q^2;q^5]_\infty[q^3;q^5]_\infty}{[q;q^5]_\infty[q^4;q^5]_\infty}$$
$$=1+\frac{q}{1+}\ \frac{q^2}{1+}\ \frac{q^3}{1+}\ \frac{q^4}{1\ +\cdots}. \qquad (5.2.21)$$

5.2.1. Continued fraction representations

In Section 5.2 we established the continued fraction representations for the ratio of two ${}_2\phi_1$'s. The success of the method depends on the existence of a suitable transformation of ${}_2\phi_1$ in terms of another ${}_2\phi_1$. In the present case we make use of a known transformation of a ${}_3\phi_2$ in terms of another ${}_3\phi_2$, both the functions having constant arguments (due to Hall, N.A.)

$$
{}_3\phi_2\left[\begin{matrix} a, & b; & c; & ef/abc \\ e, & f & & \end{matrix}\right]
= \frac{[a]_\infty [ef/ab]_\infty [ef/ac]_\infty}{[e]_\infty [f]_\infty [ef/abc]_\infty}\, {}_3\phi_2\left[\begin{matrix} e/a, & e/f, & ef/abc; & a \\ & ef/ab, & ef/ac & \end{matrix}\right]. \tag{5.2.22}
$$

Now, let us define the functions H_i and F_i as follows:

$$
H_i = \frac{1}{[e]_i [f]_i} {}_3\phi_2\left[\begin{matrix} a, & bq^i, & cq^i; & ef/abc \\ & eq^i, & fq^i & \end{matrix}\right] \tag{5.2.23}
$$

and

$$
F_i = \frac{1}{[e]_i [f]_i} {}_3\phi_2\left[\begin{matrix} aq, & bq^i, & cq^i; & ef/qabc \\ & eq^i, & fq^i & \end{matrix}\right]. \tag{5.2.24}
$$

Now,

$$
\begin{aligned}
H_i - F_i &= \sum_{n=0}^{\infty} \frac{[bq^i]_n [cq^i]_n (ef/abc)^n}{[e]_{i+n} [f]_{i+n} [q]_n} ([a]_n - [aq]_n/q^n) \\
&= -\sum_{n=0}^{\infty} \frac{[aq]_{n-1} [bq^i]_n [cq^i]_n (ef/abcq)^n (1-q^n)}{[q]_n [e]_{i+n} [f]_{i+n}} \\
&= -\frac{[aq]_n [bq^i]_{n+1} [cq^i]_{n+1} (ef/abcq)^{n+1}}{[q]_n [e]_{n+i+1} [f]_{n+i+1}} \sum_{n=0}^{\infty} \frac{[aq]_n [bq^i]_{n+1} [cq^i]_{n+1} (ef/abcq)^{n+1}}{[q]_n [e]_{n+i+1} [f]_{n+i+1}} \\
&= -(1-bq^i)(1-cq^i)(ef/abcq) F_{i+1}.
\end{aligned}
$$

Thus,

$$
H_i - F_i = -(1-bq^i)(1-cq^i)(ef/abcq) F_{i+1}. \tag{5.2.25}
$$

Now, making use of (5.2.22)to transform H_i and F_i we get

$$
H_i = \frac{[a]_\infty [efq^i/ab]_\infty [efq^i/ac]_\infty}{[e]_\infty [f]_\infty [ef/abc]_\infty} {}_3\phi_2\left[\begin{matrix} eq^i/a, & fq^i/a, & ef/abc; & a \\ & efq^i/ab, & efq^i/ac & \end{matrix}\right]
$$

and

$$
F_i = \frac{[aq]_\infty [efq^{i-1}/ab]_\infty [efq^{i-1}/ac]_\infty}{[e]_\infty [f]_\infty [ef/abcq]_\infty} {}_3\phi_2\left[\begin{matrix} eq^{i-1}/a, & fq^{i-1}/a, & ef/abcq; & aq \\ & efq^{i-1}/ab, & efq^{i-1}/ac & \end{matrix}\right]. \tag{5.2.26}
$$

Thus,

$$\begin{aligned}
H_i - &(1-a)(1-ef/abcq)F_{i+1} \\
&= \frac{[a]_\infty[efq^i/ab]_\infty[efq^i/ac]_\infty}{[e]_\infty[f]_\infty[ef/abc]_\infty} \\
&\quad \times\{[ef/abc]_n - q^n[ef/abcq]_n\} \\
&= \frac{[a]_\infty[efq^i/ab]_\infty[efq^i/ac]_\infty}{[e]_\infty[f]_\infty[ef/abc]_\infty} \\
&\quad \times\sum_{n=1}^{\infty}\frac{[eq^i/a]_n[fq^i/a]_n[ef/abc]_{n-1}a^n}{[q]_{n-1}[efq^i/ab]_n[efq^i/ac]_n} \\
&= \frac{[a]_\infty[efq^i/ab]_\infty[efq^i/ac]_\infty}{[e]_\infty[f]_\infty[ef/abc]_\infty} \\
&\quad \times\sum_{n=0}^{\infty}\frac{[eq^i/a]_{n+1}[fq^i/a]_{n+1}[ef/abc]_n a^{n+1}}{[q]_n[efq^i/ab]_{n+1}[efq^i/ac]_{n+1}} \\
&= a(1-eq^i/a)(1-fq^i/a)H_{i+1}.
\end{aligned}$$

Hence,

$$H_i-(1-a)(1-ef/abcq)F_{i+1} = a(1-eq^i/a)(1-fq^i/a)H_{i+1}. \tag{5.2.27}$$

Now, (5.2.25) and (5.2.27), respectively, yield,

$$F_i - H_i = 1 + \frac{(1-bq^i)(1-cq^i)ef}{abcqH_i/F_{i+1}} \tag{5.2.28}$$

and

$$\frac{H_i}{F_{i+1}} = (1-a)(1-ef/abcq) + \frac{a(1-eq^i/a)(1-fq^i/a)}{F_{i+1}/H_{i+1}}. \tag{5.2.29}$$

Now, iterating with the help of (5.2.28) and (5.2.29) we get, for i=0,

$$\frac{H_0}{F_0} = \frac{3\phi_2\left[\begin{matrix} aq, & b, & c; & ef/abcq \\ & e, & f & \end{matrix}\right]}{3\phi_2\left[\begin{matrix} a, & b & ,c; & ef/abc \\ & e, & f & \end{matrix}\right]}$$

$$= 1 + \frac{\eta(1-b)(1-c)}{\mu\ +}\,\frac{a(1-e/a)(1-f/a)}{1\ +}\,\frac{\eta(1-bq)(1-cq)}{\mu\ +}$$

$$+ \frac{a(1-eq/a)(1-fq/a)}{1\ +}\,\frac{\eta(1-bq^2)(1-cq^2)}{\mu\ +}$$

$$+ \frac{a(1-eq^2/a)(1-fq^2/a)}{1\ \ + \cdots} \tag{5.2.30}$$

where,

$$\eta = ef/abcq \ \text{ and } \ \mu = (1_a)(1-\eta)\cdots .$$

(5.2.30) includes scores of results as its special cases.

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5.3. Ramanujan's Theories of Theta and Elliptic Functions-IV

[*This section is based on the lectures of Professor S. Bhargava of the Department of Mathematics, University of Mysore, Manasa Gangotri, Mysore 570 006, India*]

5.3.0. Introduction

The present lectures are aimed at number theoretic applications. The lectures are a sequel to earlier lectures delivered by the author in June-July 2000, March-April 2005 SERC Schools, vide publications 31 and 32 of Centre for Mathematical Sciences, Trivandrum and Pala Campuses respectively and lecture notes by the author for March-April 2006 SERC Schools, vide publications 33 of the Centre.

5.3.1. Partition functions

We will open up this section with the definition of a partition.

Definition 5.3.1. If n is a positive integer, let $p(n)$ denote the number of unrestricted representations of n as a sum of positive integers, where representations with different orders of the same summands are not regarded as distinct. We call $p(n)$ *the partition function.*

Exercise 5.3.1. Write down all the partitions of 4,5 and 6 and show that $p(4) = 5$, $p(5) = 7$ and $p(6) = 11$.

Theorem 5.3.1. *(The generating function for $p(n)$)*

The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}$$

where $(q;q)_\infty = (1-q)(1-q^2)\cdots(1-q^n)\cdots$ and $p(0) = 1$ by convention.

Proof 5.3.1. It is sufficient to observe that the factor

$$\frac{1}{1-q^k} = \sum_{j=0}^{\infty} q^{jk} \quad \text{for } |q| < 1$$

generates the number of k's that appear in a particular partition of n. Each partition of n appears once and only once on the right side of the identity of the statement of the theorem.

Exercise 5.3.2. Show that $p_d(n)$, the number of partitions of n into distinct parts, has as its generating function

$$\sum_{n=0}^{\infty} p_d(n)q^n = (-q;q)_\infty$$

where

$$(aq;q)_\infty = (1-aq)(1-aq^2)\cdots(1-aq^n)\cdots$$

Theorem 5.3.2. *The number of partitions of a positive integer n into distinct parts equals the number of partitions of n into odd parts, denoted by* $p_0(n)$.

Proof 5.3.2. We have, from the exercise above,

$$\sum_{n=0}^{\infty} p_d(n)q^n = (-q;q)_\infty.$$

It is easy to see that

$$\text{(i)} \quad (-q;q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q)_\infty} = \frac{1}{(q;q^2)_\infty}$$

$$\text{(ii)} \quad \sum_{n=0}^{\infty} p_0(n)q^n = \frac{1}{(q;q^2)_\infty}.$$

Equating coefficients of like powers of q^n we have proved the theorem.

Exercise 5.3.3. Show that $p_d(6) = 4$ by writing down the relevant partitions. Show also that $p_0(6) = 4$ by writing down the relevant partitions.

Remark 5.3.1. Hardy and Ramanujan (1918) showed that, as $n \to \infty$,

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right),$$

that is, the ratio of the two sides tends to 1 as n tends to ∞. Proof is beyond the scope of these lectures.

Definition 5.3.2. Given positive integers n and k, let $r_k(n)$ denote the number of representations of n as a sum of k squares, where representations with different orders and different signs are counted as distinct. Conventionally, $r_k(0) = 1$.

Exercise 5.3.4. Write down all representations relevant to $r_2(2)$ and thereby show that $r_2(2) = 4$. Similarly, show that $r_2(9) = 4$ and $r_2(7) = 0$.

Definition 5.3.3. Given a positive integer n the numbers $n^2, n(n+1)/2$ and $n(3n-1)/2$ are respectively called square, triangular and pentagonal numbers.

Exercise 5.3.5. Draw pictures to justify the terminology of Definition 5.3.3.

Hint: Use the corners of a square to represent $2^2 = 4$. Then extend two of the sides of the square to draw a bigger square. Consider the three new corners and the two midpoints of the other two sides of the new square. They add up to $4 + 3 + 2 = 9$. Similarly, work with a triangle and a pentagon.

Exercise 5.3.6. Show that

$$\phi^k(q) = \sum_{n=0}^{\infty} r_k(n)q^n \text{ with } \phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

$$\psi^k(q) = \sum_{n=0}^{\infty} t_k(n)q^n \text{ with } \psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}$$

where $r_k(n)$ is the number of representations of n as sum of k squares (already defined above) and $t_k(n)$ is the number of representations of n into k triangular numbers. (With the usual convention that representations with different order are counted as distinct, for example, $t_2(7) = 2, t_2(16) = 4$.)

Exercise 5.3.7. Write down the theta function which generates pentagonal numbers. Which theta functions generate the square numbers, the triangular numbers?

Exercise 5.3.8. Show that

$$p(n) = \sum_{0<w_j\le n} (-1)^{j+1} p(n-w_j)$$

where $w_j = \frac{j(3j-1)}{2}, -\infty < j < \infty$.

Exercise 5.3.9. If $\sigma(n) = \sum d$ where the summation is taken over all divisors d of $n > 1$, show that

$$np(n) = \sum_{j=0}^{n-1} p(j)\sigma(n-j).$$

Theorem 5.3.3. *We have*

$$D_e(n) - D_0(n) = \begin{cases} (-1)^j, & \text{if } n = j(3j \pm 1)/2 \\ 0, & \text{otherwise,} \end{cases}$$

where $D_e(n)$ denotes the number of partitions of n into an even number of distinct parts and $D_0(n)$ denotes the number of partitions of n into an odd number of distinct parts.

Remark 5.3.2. The above theorem is a combinatorial version of Euler's pentagonal number theorem, namely,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

Theorem 5.3.4. *(Ramanujan's congruence). For each nonnegative integer n,*

$$p(5n+4) \equiv 0 \ (mod\ 5).$$

Proof 5.3.4. (Ramanujan):

We have

$$(q^5;q^5)_\infty \sum_{m=0}^{\infty} p(m)q^{m+1} = q\frac{(q^5;q^5)_\infty}{(q;q)_\infty} = q(q;q)_\infty^4 \frac{(q^5;q^5)_\infty}{(q;q)_\infty^5}. \tag{5.3.1}$$

Now, by the binomial theorem,

$$(q;q)_\infty^5 \equiv (q^5;q^5)_\infty (\text{mod } 5),$$

or,

$$\frac{(q^5;q^5)_\infty}{(q;q)_\infty^5} \equiv 1 (\text{mod } 5). \tag{5.3.2}$$

Now, (5.3.1) and (5.3.2) give

$$q(q;q)_\infty^4 \equiv (q^5;q^5)_\infty \sum_{m=0}^{\infty} p(m) q^{m+1} (\text{mod } 5). \tag{5.3.3}$$

Thus, from (5.3.3), to show that the $p(5n+4) \equiv 0$ (mod 5), it is enough to show that the coefficients of q^{5n+5} on the left side of (5.3.3) are multiples of 5. Now, we have

$$\begin{aligned} q(q;q)_\infty^4 &= q(q;q)_\infty (q;q)_\infty^3 \\ &= q \sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j+1)/2} \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)/2} \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} (2k+1) q^{1+\frac{j(3j+1)}{2}+\frac{k(k+1)}{2}} \end{aligned} \tag{5.3.4}$$

But,

$$2(j+1)^2 + (2k+1)^2 = 8\left\{1 + \frac{1}{2} j(3j+1) + \frac{1}{2} k(k+1)\right\} - 10j^2 - 5. \tag{5.3.5}$$

Since $2(j+1)^2 \equiv 0, 2$ or 3 (mod 5) and $(2k+1)^2 \equiv 0, 1$ or 4 (mod 5) and hence the left side of (5.3.5) is $\equiv 0$ (mod 5), iff $2(j+1)^2 \equiv 0$ (mod 5) and $(2k+1)^2 \equiv 0$ (mod 5) individually. In particular, $2k+1 \equiv 0$ (mod 5) and hence, by (5.3.4), the coefficient of $q^{5n+5}, n \geq 0$ that is $q(q;q)_\infty^4$ is a multiple of 5. The coefficient of q^5 on the right side of (5.3.3) is also a multiple of 5. This proves the theorem.

Remark 5.3.3. There are several other theorems such as Theorem (5.3.4). See Berndt (2006) for this and for an exposition of implications of Ramanujan's works within Ramanujan's repertoire.

5.3.2. Sum of squares

We have already seen in our earlier lectures that, for each positive integer n,

$$r_2(n) = 4(d_{1,4}(n) - d_{3,4}(n))$$

where $d_{j,k}(n)$ denotes the number of positive divisors d of n such that $d \equiv j \pmod{k}$.

Theorem 5.3.5. *(Sum of Four Squares): For each positive integer n, we have*

$$r_4(n) = 8 \sum d$$

where the summation is taken over all divisors d of n such that d is not divisible by 4.

Proof 5.3.5. We have the ${}_1\psi_1$- Summation of Ramanujan

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(az)_\infty (\frac{q}{az})_\infty (q)_\infty (\frac{b}{a})_\infty}{(z)_\infty (\frac{b}{az})_\infty (b)_\infty (\frac{q}{a})_\infty}. \tag{5.3.6}$$

In this, first replace q by q^2 and then put $z = e^{i\theta}$, $a = -1$ and $b = -q^2$ where θ is real. Here, $(a)_0 = 1$, $(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), n > 0$ and $(a)_\infty = (1-a)(1-aq)\cdots(1-aq^n)\cdots$, as usual.

Briefly, we get,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{(-1;q^2)_n}{(-q^2;q^2)_n} q^n \mathrm{e}^{in\theta} &= 1 + 4\sum_{n=1}^{\infty} \frac{q^n \cos n\theta}{1+q^{2n}} \\ &= \frac{(-q\mathrm{e}^{i\theta};q^2)_\infty (-q\mathrm{e}^{-i\theta};q^2)_\infty (q^2;q^2)_\infty^2}{(q\mathrm{e}^{i\theta};q^2)_\infty (q\mathrm{e}^{-i\theta};q^2)_\infty (-q^2;q^2)_\infty^2}.\end{aligned} \tag{5.3.7}$$

(Here, as usual $(a;q)_0 = 1$, $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^n), n > 0$ and $(a;q)_\infty = (1-a)(1-aq)\cdots(1-aq^n)\cdots)$. Changing θ to $\pi - \theta$ in (5.3.7) and multiplying the resulting identity with (5.3.7) we get

$$\phi^4(-q^2) = \left\{1 + 4\sum_{n=1}^{\infty} \frac{q^n \cos n\theta}{1+q^{2n}}\right\}\left\{1 + 4\sum_{n=1}^{\infty} \frac{(-q)^n \cos n\theta}{1+q^{2n}}\right\}.$$

Integrating both sides with respect to θ over $[-\pi, \pi]$ we have

$$\phi^4(q) = 1 + 8\sum_{n=1}^{\infty} \frac{q^n}{(1+(-q^n))^2} = 1 + 8\sum_{m=1}^{\infty} \frac{mq^m}{1+(-q)^m}.$$

This is simply the analytic version of the theorem.

Exercise 5.3.10. Work out the series manipulation in detail.

Remark 5.3.4. For elementary proofs of theorem similar to the one above, that is regarding sums of six squares, eight squares, sums of triangular number etc., see Berndt (2006).

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