LECTURE NOTES

SPECIAL FUNCTIONS AND FUNCTIONS OF MATRIX ARGUMENT: RECENT ADVANCES AND APPLICATIONS IN STOCHASTIC PROCESSES, STATISTICS, WAVELET ANALYSIS AND ASTROPHYSICS

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Centre for Mathematical Sciences Pala Campus,
St. Thomas College, Arunapuram P.O., Pala, Kerala 686 574, India
E-mail: cmspala@gmail.com ; mathai@math.mcgill.ca
Phone: 91+4822-216317 ; Fax: 91+4822-216317
Website: www.cmskerala.org

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SPECIAL FUNCTIONS AND FUNCTIONS OF MATRIX ARGUMENT: RECENT ADVANCES AND APPLICATIONS IN STOCHASTIC PROCESSES, STATISTICS, WAVELET ANALYSIS AND ASTROPHYSICS

LECTURE NOTES

of the

5th S.E.R.C.SCHOOL ON SPECIAL FUNCTIONS AND
FUNCTIONS OF MATRIX ARGUMENT:
RECENT ADVANCES AND APPLICATIONS IN STOCHASTIC PROCESSES,
STATISTICS, WAVELET ANALYSIS AND ASTROPHYSICS
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A.M. MATHAI

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Centre for Mathematical Sciences Pala Campus,
Arunapuram P.O., Pala-686 574, Kerala, India
Phone: 04822-216317 ; Fax: 04822-216317
Website: www.cmskerala.org
E-mail: cmspala@gmail.com

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This is publication No.34 of the Publications Series of the Centre for Mathematical Sciences (CMS). This is compiled from the lectures to be given at the 5th S.E.R.C.(Science and Engineering Research Council) School on Special Functions and Functions of Matrix Argument: Recent Advances and Applications in Stochastic Processes, Statistics, Wavelet Analysis and Astrophysics, sponsored by the Department of Science and Technology, Government of India, New Delhi, to be conducted by CMS during the period 23rd April to 25th May 2007 and to be hosted by St. Thomas College Palai. The Course Director is Dr. A.M. Mathai and the Course Co-Director is Dr. K.K Jose.

The Centre would like to thank the Department of Science and Technology, New Delhi, India, for financial assistance which made the School as well as this publication possible and Dr. B.D. Acharya, Advisor to Government of India, Mathematical Sciences Division of DST, New Delhi, for the constant encouragement and whose enthusiasm and initiative are mainly responsible for conducting this School and for bringing out this publication.

Various chapters and sections are compiled from the lecture notes supplied by the various lecturers at the School and the details are given in each chapter. The respective lecturers are solely responsible for the contents and accuracy of their notes.

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Centre for Mathematical Sciences Pala Campus,
Arunapuram P.O., Pala-686 574, Kerala, India
Phone: 04822-216317 ; Fax: 04822-216317
E-mail: cmspala@gmail.com ; mathai@math.mcgill.ca
Website: www.cmskerala.org

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PREFACE

The first S.E.R.C.(Science and Engineering Research Council) School on Special Functions was sponsored by DST (Department of Science and Technology), New Delhi, and conducted by CMS for six weeks in 1995. The second S.E.R.C. School on Special Functions and Functions of Matrix Argument was sponsored by DST, New Delhi, and conducted by CMS for five weeks during the period 29th May to 30th June 2000. In the second School, the main lectures were given by Dr. H.L. Manocha of New Delhi, Dr. S. Bhargava of Mysore, Dr. K. Srinivasa Rao and Dr. R. Jagannathan of Chennai and Dr. A.M. Mathai of Montreal, Canada. Supplementary lectures were given and problem sessions were supervised by Dr. K.S.S. Nambooripad and Dr. S.R. Chettiyar of CMS, Dr. R.N. Pillai (retired), Dr. T.S.K. Moothathu and Dr. Yageen Thomas of the University of Kerala and Dr. E. Krishnan of the University College Trivandrum. Lecture Notes were brought out as Publication No.31 of CMS soon after the School was completed. The first two Schools were conducted in Trivandrum (Thiruvananthapuram) area. Dr. A.M. Mathai was the Director and Dr. K.S.S. Nambooripad the Co-Director of these two Schools.

The third S.E.R.C. School on Special Functions and Functions of Matrix Argument: Recent Developments and Recent Applications in Statistics and Astrophysics, sponsored by DST, New Delhi, was conducted for five weeks from 14th March to 15th April 2005 by CMS at its Pala Campus, Kerala, India. This time DST, Delhi, wanted the lecture notes to be collected from the main lecturers in advance, compiled and distributed prior to the start of the School. The main lectures for the 3rd School were given by Dr. Hans J. Haubold of the Office of Outer Space Affairs of the United Nations, Dr. Serge B. Provost, Professor of Actuarial Sciences and Statistics of the University of Western Ontario, Canada, Dr. R.K. Saxena of Jodhpur, India, Dr. S. Bhargava of Mysore, India and Dr. A.M. Mathai of Montreal, Canada. Supplementary lectures were given by Dr. A. Sukumaran Nair (Chairman, CMS), Dr. K.K. Jose (Director-in-Charge, CMS Pala Campus), Dr. R.N. Pillai, Dr. Yageen Thomas, Dr. V. Seetha Lekshmi, Dr. Alice Thomas, Dr. E. Sandhya, Dr. S. Satheesh and Dr. K. Jayakumar. Problem sessions were supervised by Dr. Sebastian George and K. M. Kurian of the Department of Statistics of St.Thomas College, Palai. A special lecture sequence on Matlab was conducted by Alexander Haubold of Columbia University, U.S.A.

The 4th S.E.R.C. School on Special Functions and Functions of Matrix Argument: Recent Advances and Applications in Stochastic Processes, Statistics and Astrophysics, sponsored by DST, New Delhi, was conducted by CMS for five weeks from 6th March to 7th April 2006 at its Pala Campus. The main lectures were scheduled to be given by Dr. Hans Haubold of the Office of Outer Space Affairs of the United Nations, Dr. P.N. Rathie of the University of Brasilia, Brazil, Dr. S. Bhargava of Mysore, India, Dr. R.K. Saxena of Jodhpur, India, and Dr. A.M. Mathai of Montreal, Canada. But due to emergency Dr. Haubold and Dr. Bhargava could not reach the venue. Supplementary lectures were given by Dr. A. Sukumaran Nair (Chairman,CMS), Dr. K.S.S. Nambooripad (Director-in-Charge, CMS Trivandrum Campus), Dr. R.Y. Denis of Gorakhpur, India, Dr. N. Unnikrishnan Nair (former Vice-Chancellor of Cochin University of Science and Technology), Dr. P. Yageen Thomas and Dr. K. Jayakumar. One week was devoted to Stochastic Processes and recent advances in this area in the 4th School. Problem sessions were supervised by Dr. Joy Jacob, Dr. Sebastian George and the lecturers. Dr. A. M. Mathai has been the Director and Dr. K.K. Jose the Co-Director of the 3rd, 4th and 5th Schools.

Chapter 1 introduces elementary classical special functions. Gamma, beta, psi, zeta functions, hypergeometric functions and the associated special functions, generalizations to Meijer's G and Fox's H-functions are also examined here. Discussion is confined to basic properties and some applications. Introduction to statistical distribution theory is given here. Some recent extensions of Dirichlet integrals and Dirichlet densities are also given. A glimpse into multivariable special functions such as Appell's functions and Lauricella functions is also given. A special feature of the 4th School was a discussion of special functions as solutions of differential equations.

Chapter 2 is devoted to an introduction to Wavelet analysis. Only the basic properties are dealt with in this chapter. Chapter 3 starts with Appell's functions, then goes into fractional calculus. Saigo operators, Saigo-Maeda operators and fractional integration and differentiation of functions are also examined here. Riemann-Liouville operators, Mittag-Leffler functions and fractional differential equations are considered next. These topics are continuations of the same topics in the lecture notes of earlier Schools.

Chapter 4 is devoted to birth and death stochastic processes, continued fractions, orthogonal polynomials, followed by order statistics. The material on stochastic processes and time series had to be split into two chapters due to late arrival of parts of the notes. Chapter 5 starts with basic or q-hypergeometric functions, then goes into Ramanujan's elliptic functions and theta functions. These sections are continuations from the lecture notes of the previous SERC Schools.

Chapter 6 is on time series and general stochastic processes and chapter 7 deals with functions of matrix argument. In both these chapters some new and current results are included. A new topic on partial differential equations and their applications is included as chapter 8. Chapter 9 deals with applications in astrophysics problems. Some new and current results are also added in this chapter.

Several people have contributed enormously for the success of this SERC School and for making this publication possible. Dr. B.D. Acharya and Dr. Ashok K. Singh of DST, New Delhi are especially thanked for sanctioning the project and for getting the funds released on time. All the lecturers who supplied their lecture notes, namely, Dr. H.J. Haubold, Dr. D.V. Pai, Dr. S. Bhargava, Dr. R.K. Saxena, Dr. A.M. Mathai, Dr. K.K. Jose, Dr. R.Y. Denis, Dr. P.R. Parthasarathy, Dr. P. Yageen Thomas, Dr. J.J. Xu are thanked for their contributions to this Lecture Notes. Rev. Dr. Mathew John K, Pricipal and Rev. Fr. N. Joseph, Bursar, St. Thomas College Palai are especially thanked for providing all facilities of the college and its guest houses for the 5th SERC School. All the faculty and postgraduate students in various departments at St. Thomas College Palai, especially the faculty and students of the departments of Statistics, Mathematics and Physics, spent hours of their time for making the School a grand success. The Director of the Pastoral Institute, the international guest house of the College is thanked for making the stay of the participants comfortable. The college canteen authorities are thanked for

the timely supply of good food for the participants. Soby Haridas, our Office Manager, set the manuscript by using LaTeX. Soby Haridas and the Project Assistant Sini Devassy are thanked for their selfless efforts. Our sincere thanks go to each and everyone, who directely or indirectely helped to make the 5th SERC School and this publication a grand success.

Pala, Kerala, India — A.M. Mathai
20th February, 2007

CONTENTS

GLOSSARY OF SYMBOLS

$(a)_m$	Pochammer symbol	Notation 1.0.1	1
$n!$	factorial n or n factorial	Notation 1.0.2	2
$\binom{n}{r}$	number of combinations	Notation 1.0.3	2
$\Gamma(z)$	gamma function	Notation 1.1.1	3
γ	Euler's constant	Notation 1.1.2	4
$R(\cdot)$	real part of $(\cdot)$	Definition 1.1.5	4
$B_k^{(a)}(x)$	generalized Bernoulli polynomial	Notation 1.2.1.	7
$B_k(x)$	Bernoulli polynomial	Notation 1.2.1.	7
B_k	Bernoulli number	Notation 1.2.1.	7
$\psi(z)$	psi function	Notation 1.3.1	9
$\zeta(\rho)$	Riemann zeta function	Notation 1.3.2	10
$\zeta(\rho, a)$	generalized zeta function	Notation 1.3.2	10
$\wedge$	wedge product	Section 1.4	12
$x\|y$	x given y	Section 1.4.2	15
$E(\cdot)$	expected value of $(\cdot)$	Definition 1.4.3	17
$M_j(t_1, \cdots, t_k)$	moment generating function	Section 1.4.4	18
$L_f(\cdot)$	Laplace transform of f	Section 1.4.4	18
$\Phi_f(\cdot)$	Fourier transform of f	Section 1.4.4	18
$\text{Var}(\cdot)$	variance of $(\cdot)$	Exercise 1.4.10	20
$B(\alpha, \beta)$	beta function	Section 1.5.1	21
$\sim$	distributed as	Section 1.5	26
$\text{Cov}(\cdot, \cdot)$	covariance	Exercise 1.5.5	26
${}_pF_q(\cdot)$	hypergeometric function	Section 1.7	34
$\gamma(a, x), \Gamma(a, x)$	incomplete gamma function	Exercise 1.7.1	37
$B_x(\alpha, \beta), I_x(\alpha, \beta)$	incomplete beta function	Exercise 1.7.2	37
$M_{\mu,\nu}(Z)$	Whittaker function	Exercise 1.7.3	38
$W_{\mu,\nu}(Z)$	Whittaker function	Exercise 1.7.4	38
$I_\nu(z), J_\nu(z)$	Bessel function	Exercise 1.7.5.	38
$Y_\nu(z), K_\nu(z)$	Bessel function	Exercise 1.7.5.	38
$G_{p,q}^{m,n}(z)$	Majer's G-function	Section 1.8	41
$H_{p,q}^{m,n}(z)$	H-function	Section 1.9	49

$L^2(0,2\sigma)$	space of equivalence class	Section 2.1	54
(f,g)	inner product	Section 2.1	54
$\|f\|$	norm	Section 2.1	54
$\psi_{a,b}(t)$	wavelets	Definition 2.1.3	58
$(W_\psi f)(a,b)$	wavelet transform	Definition 2.1.3	58
$\psi^H(t)$	Haar function	Section 2.2	59
F_3	Appell function	Section 3.1	79
$M\{f(x):s\}$	Mellin transform	Section 3.1.2	80
$I_{0+}^{\alpha,\beta,\eta}, I_{-}^{\alpha,\beta,\eta}$	Saigo operators	Section 3.2.1	82
$D_{0+}^{\alpha,\beta,\eta}, D_{-}^{\alpha,\beta,\eta}$	Saigo operators	Section 3.2.1	82
$I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}, I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$	Saigo-Maeda operators	Section 3.2.1	82
$D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}, D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$	Saigo-Maeda operators	Section 3.2.1	82
$E_\alpha(z), E_{\alpha,\beta}(z)$	Mittag-Leffler functions	Section 3.4	94
$E_{\alpha,\beta}^{\gamma}(z)$	Mittag-Leffler function	Section 3.4	94
$\phi_3(\beta,\gamma;x,y)$	confluent hypergeometric function of two variables	Notation 3.4.4.	95
$\phi(\alpha,\beta;z)$	confluent hypergeometric function of one variable	Notation 3.4.5.	95
I_x^α, D_x^α	Riemann-Liouville operators	Notations 3.4.6, 3.4.7	95
${}_0^C D_t^\alpha$	Caputo operator	Section 3.4.2	100
BDP	birth and death processes	Section 4.1	109
$P_{ij}(t)$	transition probability	Section 4.2	112
CF	continued fraction	Section 4.3	119
RSS	ranked-set sampling	Section 4.5	137
$\mathrm{Var}(\cdot)$	variance	Section 4.5	140
$\mathrm{Cov}(\cdot,\cdot)$	covariance	Section 4.5	140
$\mathrm{Corr}(\cdot,\cdot)$	correlation	Remark 4.5.4	153
${}_pF_q(\cdot)$	hypergeometric series	Section 5.0	157
$[\alpha]_q$	ratio symbol	Equation (5.0.3)	158
$[a;q]_n$	a product symbol	Equation (5.0.6)	159
$r\phi_s(\cdot)$	hypergeometric series	Equation (5.0.7)	159
$(q;q)_\infty$	product notation	Section 5.3.1	172

AR(q)	autoregressive model	Section 6.3.3	195
ARMA(q,p)	autoregressive moving average	Section 6.3.4	195
ARIMA(q,r,p)	autoregressive integrated model	Section 6.3.5	196
GMLD(α,β)	geometrically infinitely divisible	Section 6.4.2	202
GGMLD(α,β)	Mettag-Leffler distribution	Section 6.4.4	205
MGGEP	Pareto model	Section 6.5.2	212
$\|(\cdot)\|$	determinant of (.)	Section 7.0	225
$\mathrm{tr}(\cdot)$	trace of (.)	Section 7.0	225
$\mathrm{d}X$	wedge product of differentials	Section 7.0	225
X'	transpose of the matrix X	Section 7.0	225
$X>0$	positive definite X	Section 7.0	225
$\Gamma_p(\alpha)$	real matrix-variate gamma	Notation 7.2.1	237
$(\mathrm{d}X)$	matrix of differentials	Proof 7.2.2	238
$V_{p,n}$	Stiefel manifold	Remark 7.3.1	244
O_p	orthogonal group	Remark 7.3.1	244
$B_p(\alpha,\beta)$	real matrix-variate beta	Definition 7.4.1	251
$L_f(T^*)$	Laplace transform	Section 7.5	257
${}_pF_q(\cdot)$	hypergeometric function	Section 7.6.1	263
$C_K(Z)$	zonal polynomial	Section 7.6.2	264
$\Gamma_p(\alpha,K)$	gamma product	Section 7.6.2	265
$M_\alpha(f)$	M-transform	Section 7.6.3	267
$W(\cdot)$	Whittaker function	Exercise 7.6.10	271
q_{sen}	q-sensitivity	Section 9.2	329
q_{stat}	q-stationary	Section 9.2	329
q_{rel}	q-relaxation	Section 9.2	329
$M_{k,\alpha}(P), H_{k,\alpha}(P), T_{k,\alpha}(P)$, $M_{k,\alpha}(P)$	generalized entropies	Section 9.5	344