CHAPTER 1

PRELIMINARIES OF SPECIAL FUNCTIONS AND STATISTICAL DISTRIBUTIONS

[*This chapter is based on the lectures of Professor A.M. Mathai of McGill University, Canada (Director of the 3rd SERC School).*]

1.0. Introduction

A very brief outline of some elementary special functions and statistical distributions is given here to make the notes self-contained. Details are available from the following sources, which are accessible to the participants of the 3^{rd} SERC School:

(1) *Notes of the* 2^{nd} *SERC School*. (Publication No 31 of the Centre for Mathematical Sciences (CMS)), 2000.

(2) Mathai, A.M. (1993). *"A Handbook of Generalized Special Functions for Statistical and Physical Sciences"*, Oxford University Press, Oxford, U.K.

(3) Mathai, A.M. and Saxena, R.K. (1978). *"The H-Function with Applications in Statistics and Other Disciplines"*, Wiley Halsted, New York.

(4) Mathai, A.M. and Saxena (1973). *"Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences"*, Lecture Notes No 348, Springer-Verlag, Heidelberg.

Notation 1.0.1. Pochammer Symbol

$$(a)_m = a(a+1)\cdots(a+m-1),\ (a)_0 = 1,\ a \neq 0. \tag{1.0.1}$$

For example,

$$\left(-\frac{2}{3}\right)_2 = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}+1\right) = -\frac{2}{9}; \quad (-3)_3 = (-3)(-2)(-1) = -6;$$
$$(-3)_5 = (-3)(-2)(-1)(0)(1) = 0; \quad \left(\frac{1}{2}\right)_3 = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right) = \frac{15}{8}.$$

Notation 1.0.2. **Factorial *n* or *n* factorial**

$$n! = (1)(2)\cdots(n),\ 0! = 1 \text{ (convention).} \tag{1.0.2}$$

For example,

$$3! = (1)(2)(3) = 6;\quad \frac{2}{3}! = \text{not defined}$$
$$(-2)! = \text{not defined};\quad 1! = 1;\quad 0! = 1\quad \text{(convention).}$$

Notation 1.0.3. **Number of Combinations of *n* taken *r* at a time**

$$\binom{n}{r} = \text{number of subsets of } r \text{ distinct objects from a set of } n \text{ distinct objects} \tag{1.0.3}$$
$$= \frac{n(n-1)\cdots(n-(r-1))}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!},\quad 0 \le r \le n.$$

For example,

$$\binom{3}{1} = \frac{3}{1!} = 3;\ \binom{3}{2} = \frac{(3)(2)}{2!} = 3;\ \binom{n}{1} = \frac{n}{1!} = n;$$
$$\binom{n}{n-1} = \frac{n(n-1)\cdots(n-(n-1))}{(n-1)!} = n \Longrightarrow \binom{n}{1} = \binom{n}{n-1};$$
$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1;\ \binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = 1 \Longrightarrow \binom{n}{0} = \binom{n}{n};$$
$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)![n-(n-r)]!} = \binom{n}{n-r};\ \binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1};$$
$$\binom{-2}{3} = \text{ not defined as a combination};\ \binom{1/3}{2} = \text{not defined as a number of combinations.}$$

But if $\binom{n}{r}$ is not treated as a number of combinations but defined in terms of Pochammer symbol as

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{(-1)^r(-n)(-n+1)\cdots(-n+r-1)}{r!}$$
$$= \frac{(-1)^r(-n)_r}{r!} \tag{1.0.4}$$

then

$$\binom{-2/3}{2} = \frac{(-1)^2}{2!}\left(\frac{2}{3}\right)\left(\frac{2}{3}+1\right) = \frac{5}{9}; \ \binom{1/2}{3} = \frac{(-1)^3}{3!}\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}+2\right) = \frac{1}{16}.$$

Note that

$$(a)_{m+n} = (a)_m(a+m)_n = (a)_n(a+n)_m. \tag{1.0.5}$$

1.1. Gamma and Related Functions

Notation 1.1.1. : $\Gamma(z) =$ **gamma** z

A gamma function is defined in many ways. Some of the definitions, along with the necessary conditions are the following:

Definition 1.1.1.

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)}, \quad z \neq 0, -1, -2, \cdots \tag{1.1.1}$$

Definition 1.1.2.

$$\Gamma(z) = z^{-1}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}. \tag{1.1.2}$$

Definition 1.1.3.

$$\frac{1}{\Gamma(z)} = z\lim_{n\to\infty}\left\{n^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)\right\}. \tag{1.1.3}$$

Definition 1.1.4.

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left[(1+\frac{z}{n})e^{-z/n}\right] \tag{1.1.4}$$

where γ is the Euler's constant.

Notation 1.1.2.

$$\gamma = \lim_{n\to\infty}\left\{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right\} \approx 0.577215664901532860606512. \tag{1.1.5}$$

Definition 1.1.5.

$$\Gamma(z) = p^z \int_0^\infty t^{z-1} \mathrm{e}^{-\mathrm{pt}} \mathrm{dt}, \Re(\mathrm{p}) > 0, \Re(\mathrm{z}) > 0 \tag{1.1.6}$$

where $\Re(.)$ denotes the real part of (.).

Definition 1.1.6.

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-z} \mathrm{e}^t \mathrm{d}t,\ c > 0, \Re(z) > 0,\ i = \sqrt{-1} \tag{1.1.7}$$

where π is the mathematical constant,

$$\pi \approx 3.14159265358979323846264 3.$$

Thus, from the Laplace representation in (1.1.6) we have an integral representation

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} \mathrm{d}t,\ \Re(z) > 0. \tag{1.1.8}$$

In general, $\Gamma(z)$ exists for all values of z, positive or negative, except at the points $z = 0, -1, -2, \cdots$. These are the poles of $\Gamma(z)$. But for the integral representation in (1.1.8) to hold the real part of z must be positive. Thus, for example,

$\Gamma(5)$ exists; $\Gamma(-\frac{1}{2})$ exists; $\Gamma(0)$ does not exist; $\Gamma(-3)$ does not exist.

It is not difficult to show that

$$\Gamma(z) = (z-1)\Gamma(z-1) = (z-1)(z-2)\cdots(z-r)\Gamma(z-r) \tag{1.1.9}$$

when $\Gamma(z)$ and $\Gamma(z-r)$ are defined. It is easily established from the integral representation in (1.1.8) by integrating by parts. The property holds for other definitions also. Thus, for example,

$$\begin{aligned}\Gamma(n) &= (n-1)(n-2)\cdots 1\Gamma(1) \text{ but } \Gamma(1) = 1\\ &= (n-1)! \text{ for } n = 1, 2, \cdots \end{aligned} \tag{1.1.10}$$

$$\Gamma\Big(\frac{1}{2}\Big) = \Big(\frac{1}{2} - 1\Big)\Gamma\Big(\frac{1}{2} - 1\Big) = -\frac{1}{2}\Gamma\Big(-\frac{1}{2}\Big) \Rightarrow \Gamma\Big(-\frac{1}{2}\Big) = -2\Gamma\Big(\frac{1}{2}\Big).$$

$$\Gamma\Big(\frac{7}{2}\Big) = \Big(\frac{5}{2}\Big)\Big(\frac{3}{2}\Big)\Big(\frac{1}{2}\Big)\Gamma\Big(\frac{1}{2}\Big) = \frac{15}{8}\Gamma\Big(\frac{1}{2}\Big).$$

By using the property in (1.1.9) we can reduce any gamma function

$$\Gamma(z) = (\text{a few factors})\ \Gamma(\alpha), 0 < \alpha \leq 1 \tag{1.1.11}$$

and $\Gamma(\alpha)$ for $0 < \alpha \leq 1$ is extensively tabulated. For computational purposes one can use (1.1.11) and the extensive numerical tables for $\Gamma(\alpha), 0 < \alpha \leq 1$. It can also be shown that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. The proof is simple in terms of the integral representations in (1.1.8). Consider

$$\begin{aligned}\left[\Gamma\left(\frac{1}{2}\right)\right]^2 &= \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \left[\int_0^\infty x^{\frac{1}{2}-1}\mathrm{e}^{-x}\mathrm{d}x\right]\left[\int_0^\infty y^{\frac{1}{2}-1}\mathrm{e}^{-y}\mathrm{d}y\right]\\ &= \int_0^\infty\int_0^\infty x^{\frac{1}{2}-1}y^{\frac{1}{2}-1}\mathrm{e}^{-(\mathrm{x}+\mathrm{y})}\mathrm{dxdy}.\end{aligned}$$

Put $x = r\cos^2\theta$, $y = r\sin^2\theta$, $0 \leq r < \infty$, $0 \leq \theta \leq \frac{\pi}{2}$, the Jacobian is $2r\sin\theta\cos\theta$, integrate out r and θ to see that the right side reduces to π, and hence the result. Therefore

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{1.1.12}$$

Also

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{1.1.13}$$

when the gammas are defined.

1.1.1. Multiplication formula for a gamma function

$$\Gamma(mz) = (2\pi)^{\frac{1-m}{2}} m^{mz-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)\cdots\Gamma\left(z+\frac{m-1}{m}\right),\ m = 1, 2, \cdots \tag{1.1.14}$$

For example,

$$\Gamma(2z) = (2\pi)^{\frac{1-2}{2}} 2^{2z-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \pi^{-\frac{1}{2}} z^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

is known as the *duplication formula* for gamma functions. For example,

$$\begin{aligned}1 = \Gamma(1) = \Gamma\left[2\left(\frac{1}{2}\right)\right] &= \pi^{-\frac{1}{2}}2^{1-1}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\right) = \pi^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)\Gamma(1) \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.\\ 1 = \Gamma\left[3\left(\frac{1}{3}\right)\right] &= (2\pi)^{\frac{1-3}{2}}3^{1-\frac{1}{2}}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma(1) \Rightarrow \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}.\end{aligned}$$

By using the infinite product definitions for trigonometric functions we can establish the following results:

$$\Gamma(z)\Gamma(1-z) = \pi \ \operatorname{cosec} \pi z; \tag{1.1.15}$$

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z} \operatorname{cosec} \pi z; \tag{1.1.16}$$

$$\Gamma\Big(\frac{1}{2}+z\Big)\Gamma\Big(\frac{1}{2}-z\Big) = \pi \ \sec \pi z. \tag{1.1.17}$$

Exercises 1.1.

1.1.1. Evaluate the following whenever they exist.
(a) $\left(-\frac{3}{2}\right)_3$; (b) $(-3)_4$; (c) $(1)_n$; (d) $(0)_2$.

1.1.2. Evaluate the following, interpreting as the number of combinations, whenever they exist.
(a) $\binom{2/3}{5}$; (b) $\binom{-2}{3}$; (c) $\binom{2}{3}$; (d) $\binom{4}{2}$; (e) $\binom{100}{4}$.

1.1.3. An M.Sc Mathematics class has 5 boys and 9 girls. A committee of 4 persons is to be chosen, consisting of 2 boys and 2 girls. (a) How many total choices are there? (b) How many choices are there if there is no restriction on the number of boys and girls in the committee?

1.1.4. Prove that definitions 1.1.3 and 1.1.4 are one and the same.

1.1.5. Evaluate the following in terms of $\Gamma(\alpha), 0 < \alpha \le 1$.
(a) $\Gamma(-\frac{5}{2})$; (b) $\Gamma(-\frac{3}{4})$; (c) $\Gamma(\frac{7}{2})$; (d) $\Gamma(8)$.

1.1.6. Evaluate the following:
(a) $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})$; (b) $\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})$.

1.1.7. Show that $\Gamma(\frac{1}{6})\Gamma(\frac{5}{6}) = 2\pi$.

1.1.8. Show that $z\Gamma(z) = \Gamma(z+1)$ by using definition 1.1.1.

1.1.9. Show that $z\Gamma(z) = \Gamma(z+1)$ by using definition 1.1.2.

1.1.10. Show that $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-z}\left(1+\frac{z}{n}\right) = \lim_{n\to\infty} n^{-z} \prod_{k=1}^{n}\left(1+\frac{z}{k}\right)$.

1.2. The Psi Function

Notation 1.2.1. $\psi(z)$: **psi z**

Definition 1.2.1.

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{dz}} \ln \Gamma(z) = \frac{1}{\Gamma(z)} \frac{\mathrm{d}}{\mathrm{dz}} \Gamma(z), \ln \Gamma(z) = \int_1^z \psi(x) \mathrm{d}x \ . \tag{1.2.1}$$

It is the logarithmic derivative of the gamma function. For example,

$$\psi(z) = \frac{1}{z-1} + \frac{1}{z-2} + \cdots + \frac{1}{z-r} + \psi(z-r). \tag{1.2.2}$$

By using the various definitions and properties of gamma functions we have the following properties for a psi function:

$$\psi(z) = -\gamma - \frac{1}{z} + z \sum_{k=1}^{\infty} \frac{1}{k(z+k)} \tag{1.2.3}$$

$$\psi(z) = -\gamma + (z-1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(z+k)} \tag{1.2.4}$$

$$\psi(1) = -\gamma \tag{1.2.5}$$

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2 \ln 2 \tag{1.2.6}$$

$$\psi(z) - \psi(1-z) = -\pi \cot \pi z \tag{1.2.7}$$

$$\psi\left(\frac{1}{2} + z\right) - \psi\left(\frac{1}{2} - z\right) = \pi \tan \pi z \ . \tag{1.2.8}$$

Successive derivations of $\psi(z)$ give generalized zeta functions.

1.2.1. Generalized zeta function

Notation 1.2.2.

$\zeta(\rho, a)$: generalized zeta function
$\zeta(\rho)$: Riemann zeta function

Definition 1.2.2.

$$\zeta(\rho, a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^{\rho}},\ \Re(\rho) > 1,\ a \neq 0, -1, -2, \cdots \tag{1.2.9}$$

$$\zeta(\rho) = \sum_{k=1}^{\infty} \frac{1}{k^{\rho}},\ \Re(\rho) > 1\ . \tag{1.2.10}$$

For $\rho \leq 1$ the series is divergent. From (1.2.4), by successive differentiation, we have the following:

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} \ln \Gamma(z) = \frac{\mathrm{d}}{\mathrm{d}z}\psi(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2} = \zeta(2, z). \tag{1.2.11}$$

$$\begin{aligned} \frac{\mathrm{d}^r}{\mathrm{d}z^r} \ln \Gamma(z) = \frac{\mathrm{d}^{r-1}}{\mathrm{d}z^{r-1}}\psi(z) &= \begin{cases} \psi(z), & \text{for } r = 1 \\ (-1)^r (r-1)! \zeta(r, z), & \text{for } r \geq 2 \end{cases} \\ &= (-1)^r (r-1)! \sum_{k=0}^{\infty} \frac{1}{(z+k)^r}\ . \end{aligned} \tag{1.2.12}$$

A few explicit forms are the following:

$$\zeta(2) = \zeta(2, 1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{1.2.13}$$

$$\zeta(4) = \zeta(4, 1) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}. \tag{1.2.14}$$

$$\zeta(2k) = \zeta(2k, 1) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!} B_{2k}, \tag{1.2.15}$$

where B_{2k} is a Bernoulli number. For details of generalized Bernoulli polynomials, Bernoulli polynomials and Bernoulli numbers see Mathai (1993).

1.2.2. An asymptotic formula for gamma functions

For $|z| \to \infty$ and α a bounded quantity, it can be shown that

$$\ln\Gamma(z+\alpha) = \frac{1}{2}\ln(2\pi) + (z+\alpha-\frac{1}{2})\ln z - z$$
$$+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}B_{k+1}(\alpha)}{k(k+1)z^k}, |\arg\,(z+\alpha)| \le \pi-\epsilon,\ \epsilon>0 \qquad (1.2.16)$$

where $B_{k+1}(\alpha)$ is a Bernoulli polynomial. The first part of the formula in (1.2.16) gives *Stirling's approximation:*

$$\Gamma(z+\alpha) \approx (2\pi)^{\frac{1}{2}} z^{z+\alpha-\frac{1}{2}} \mathrm{e}^{-z}\ . \qquad (1.2.17)$$

For example, for $|z| \to \infty$, α and β bounded quantities,

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\alpha+\beta)} \approx \frac{(2\pi)^{\frac{1}{2}} z^{z+\alpha-\frac{1}{2}} \mathrm{e}^{-z}}{(2\pi)^{\frac{1}{2}} z^{z+\alpha+\beta-\frac{1}{2}} \mathrm{e}^{-z}} = z^{-\beta}\ . \qquad (1.2.18)$$

Exercises 1.2.

1.2.1. Prove formula (1.2.4) by using (1.1.9).

1.2.2. Prove formula (1.2.3).

1.2.3. Prove formula (1.2.6) by using the duplication formula for gamma functions.

1.2.4. Show that

$$\psi(1+n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma\ .$$

1.2.5. Evaluate $\psi(-\frac{3}{2})$.

1.2.6. Evaluate $\psi(5)$.

1.2.7. If $\ln\Gamma(z+1) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$ evaluate a_n, $n = 0, 1, 2, \cdots$

1.2.8. Show that $\zeta(k, \frac{1}{2}) = (2^k - 1)\zeta(k)$.

1.2.9. show that $\zeta(k, -\frac{3}{2}) = (-1)^k\Big(2^k\Big)\Big[1 + \frac{1}{3^k}\Big] + \zeta(k, \frac{1}{2})$.

1.2.10. Show that

$$\zeta\Big(k, z - \frac{2r+1}{2}\Big) = \frac{1}{\big(z-\frac{1}{2}\big)^k} + \cdots + \frac{1}{\big(z-\frac{2r+1}{2}\big)^k} + \zeta\Big(k, z+\frac{1}{2}\Big), r = 0, 1, \cdots, \quad k = 2, 3, \cdots$$

1.3. Essentials of Statistical Distribution Theory

The mathematical aspects of statistical distributions will be defined and discussed here. We will not be dealing with random variables, discrete probability functions, mixed situations etc here. Hence density functions defined on a continuum of points in the real case will be considered here.

Let $f(x_1, \cdots, x_k)$ be a non-negative integrable scalar function with the total integral unity in the real scalar variables $x_1, \cdots, x_k$.

Definition 1.3.1. A density function.

If $f(x_1, \cdots, x_k)$ satisfies the following conditions:

(i) $f(x_1, \cdots, x_k) \geq 0$ for all $x_1, \cdots, x_k$,
(ii) $\int_{x_1}, \cdots, \int_{x_k} f(x_1, \cdots, x_k)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k = 1$

then $f(x_1, \cdots, x_k)$ is called a joint density function of the real scalar random variables $x_1, \cdots, x_k$ where $\wedge$ denotes the wedge product or skew symmetric product of differentials.

Example 1.3.1. Check whether the following are density functions:

(1) $f_1(x) = \frac{1}{\theta}, 0 \leq x \leq \theta$ and $f_1(x) = 0$ elsewhere;

(2) $f_2(x) = \frac{1}{x}, 1 \leq x \leq \infty$ and $f_2(x) = 0$ elsewhere;

(3) $f_3(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 \leq x \leq 2 \end{cases}$ and zero elsewhere;

(4) $f_4(x_1, x_2) = 2, 0 \le x_1 \le x_2 \le 1$ and zero elsewhere;

(5) $f_5(x_1, \cdots, x_k) = \mathrm{e}^{-(x_1+\cdots+x_k)}, 0 \le x_j < \infty, j = 1, \cdots, k$ and zero elsewhere.

Solutions: Obviously the functions in (1) to (5) are non-negative and hence we need to check the second condition only.

(1)

$$\int_{-\infty}^{\infty} f_1(x)\mathrm{d}x = 0 + \int_0^{\theta} \frac{1}{\theta}\mathrm{d}x = \left[\frac{x}{\theta}\right]_0^{\theta} = 1.$$

Hence $f_1(x)$ is a density. This is known as a *uniform density* or the random variable x is said to be uniformly distributed over the closed interval $[0, \theta], \theta > 0$, where θ is an unknown constant. Unknown constants in a density are called *parameters*. Hence θ is a parameter here.

(2)

$$\int_{-\infty}^{\infty} f_2(x)\mathrm{d}x = 0 + \int_1^{\infty} \frac{1}{x}\mathrm{d}x = \left[\ln x\right]_1^{\infty} = \infty.$$

The integral does not converge to 1. Hence $f_2(x)$ is not a density.

(3)

$$\int_{-\infty}^{\infty} f_3(x)\mathrm{d}x = 0 + \int_0^1 x\mathrm{d}x + \int_1^2 (2-x)\mathrm{d}x = \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 = 1.$$

Hence $f_3(x)$ is a density function.

(4)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_4(x_1, x_2)\mathrm{d}x_1 \wedge \mathrm{d}x_2 = 0 + \int_{x_2=0}^{1}\left[\int_{x_1=0}^{x_2} 2\mathrm{d}x_1\right]\mathrm{d}x_2 = \int_{x_2=0}^{1} 2x_2\mathrm{d}x_2 = 1.$$

Note that the region of integration is either

$$\{(x_1, x_2)|0 \le x_1 \le x_2 \text{ and } 0 \le x_2 \le 1\} \quad \text{or} \quad \{(x_1, x_2)|x_1 \le x_2 \le 1 \text{ and } 0 \le x_1 \le 1\}.$$

We may use either of these. Thus, $f_4(x_1, x_2)$ is a joint density function of the real scalar random variables x_1 and x_2.

(5)

$$\begin{aligned}
&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_5(x_1,\cdots,x_k)\mathrm{d}x_1\wedge\cdots\wedge \mathrm{d}x_k\\
&= 0+\int_0^{\infty}\cdots\int_0^{\infty} \mathrm{e}^{-(x_1+\cdots+x_k)}\mathrm{d}x_1\wedge\cdots\wedge \mathrm{d}x_k\\
&= \prod_{j=1}^{k}\int_0^{\infty}\mathrm{e}^{-x_j}\mathrm{d}x_j = \prod_{j=1}^{k}[-\mathrm{e}^{-x_j}]_0^{\infty} = 1.
\end{aligned}$$

Hence $f_5(x_1,\cdots,x_k)$ is a joint density function of $x_1,\cdots,x_k$.

1.3.1. The marginal and conditional densities

If $f(x_1,\cdots,x_k)$ is a joint density function of $x_1,\cdots,x_k$ then the density function of any subset of these variables, say for example, $x_1,\cdots,x_r, r\le k$, is available by integrating out the other variables. The density thus obtained is called the marginal density of that subset. For example, the marginal density of x_1 is available by integrating out $x_2,\cdots,x_k$ from $f(x_1,\cdots,x_k)$.

Example 1.3.2. Evaluate the marginal densities of the individual variables in (4) and (5) of Example 1.3.1.

Solutions: Integrating out x_1 in (4), observing that x_1 goes from 0 to x_2, we obtain the marginal density of x_2, denoted by $g_2(x_2)$. That is,

$$g_2(x_2)=\int_{x_1=0}^{x_2}2\mathrm{d}x=\begin{cases}2x_2, 0\le x_2\le 1\\ 0,\quad \text{elsewhere.}\end{cases} \tag{1.3.1}$$

Similarly, observing that x_2 goes from x_1 to 1, the marginal density of x_1, denoted by $g_1(x_1)$, is available as the following:

$$g_1(x_1)=\int_{x_2=x_1}^{1}2\mathrm{d}x_2=\begin{cases}2(1-x_1), 0\le x_1\le 1\\ 0,\quad \text{elsewhere.}\end{cases} \tag{1.3.2}$$

Here $g_1(x_1)$ and $g_2(x_2)$ are the marginal densities of x_1 and x_2 respectively. In (5) of Example 1.3.1 the variables are separated and the integral over x_j gives

$$\int_0^{\infty}\mathrm{e}^{-x_j}\mathrm{d}x_j = 1., i=1,\cdots,k.$$

Hence the marginal density of of x_j, denoted by $h_j(x_j)$, is given by,

$$h_j(x_j) = \begin{cases} e^{-x_j}, 0 \le x_j < \infty \\ 0, \quad \text{elsewhere} \quad j = 1, 2, \cdots, k. \end{cases} \tag{1.3.3}$$

One interesting property may be noted from (5) of Example 1.3.1. Here the joint density is the product of the marginal densities whereas in (4) of Example 1.3.1 the joint density is not equal to the product of marginal densities.

1.3.2. Conditional densities and statistical independence

Let $f(x_1, \cdots, x_k)$ be the joint density of the real scalar random variables $x_1, \cdots, x_k$. Consider two non-overlapping subsets of random variables, for example $\{x_1, \cdots, x_r\}$, $\{x_{r+1}, \cdots, x_k\}, r < k$, the two subsets need not exhaust the whole set. Let $f_1(x_1, \cdots, x_r)$ and $f_2(x_{r+1}, \cdots, x_k)$ be the marginal densities of these mutually exclusive subsets. Then the conditional density of the first subset given the second subset, denoted by $g(x_1, \cdots, x_r|x_{r+1}, \cdots, x_k)$ is defined as

$$g(x_1, \cdots, x_r|x_{r+1}, \cdots, x_k) = \frac{f(x_1, \cdots, x_k)}{f_2(x_{r+1}, \cdots, x_k)} \tag{1.3.4}$$

for $f_2(x_{r+1}, \cdots, x_k) \neq 0$ at the given points for $x_{r+1}, \cdots, x_k$.

Example 1.3.3. Evaluate the conditional density of x_1 given $x_2 = \frac{1}{3}$ in (4) of Example 1.3.1.

Solution: The marginal densities are available from (1.3.2) and (1.3.1) respectively. Hence the conditional density of x_1 given $x_2 = \frac{1}{3}$, denoted by $g(x_1|x_2 = \frac{1}{3})$ is given by the following:

$$g(x_1|x_2 = \frac{1}{3}) = \frac{\text{Joint density of } x_1 \text{ and } x_2}{\text{Marginal density of } x_2}, \text{ evaluated at } x_2 = \frac{1}{3}$$
$$= \frac{2}{2x_2}|_{x_2} = \frac{1}{3} = 3.$$

But in the joint density x_1 goes from 0 to x_2. Hence

$$g(x_1|x_2 = \frac{1}{3}) = \begin{cases} 3, 0 \le x_1 \le \frac{1}{3} \\ 0, \quad \text{elsewhere.} \end{cases}$$

Note that here the conditional density of x_1 depends upon the condition on x_2.

Example 1.3.4. Evaluate the conditional density of x_1 given $x_2 = a_2, \cdots, x_k = a_k$ in (5) of Example 1.3.1.

Solution: The marginal densities are available from (1.3.3). Hence the marginal joint density of $x_2, \cdots, x_k$ is $\mathrm{e}^{-(x_2,\cdots,x_k)}, 0 \le x_j < \infty, j = 2, \cdots, k$ and zero elsewhere. The conditional density of x_1 given $x_2, \cdots, x_k$, denoted by $g(x_1|x_2 \cdots, x_k)$ is then

$$g(x_1|x_2, \cdots, x_k) = \frac{\text{Joint density of } x_1, \ \cdots, x_k}{\text{Marginal density of } x_2, \cdots, x_k} = \frac{\mathrm{e}^{-(x_1+\cdots+x_k)}}{\mathrm{e}^{-(x_2+\cdots+x_k)}} = \mathrm{e}^{-x_1}$$

$$= \begin{cases} \mathrm{e}^{-x_1}, 0 \le x_1 < \infty \\ 0, \quad \text{elsewhere.} \end{cases}$$

Here, whatever be the values at which $x_2, \cdots, x_k$ are fixed or given, that has no relevance to the density of x_1. Further, note that in this case the joint density is the product of marginal densities. If such a thing happens then the variables are said to be independent or statistically independently distributed.

Definition 1.3.2. Statistical independence.

If the joint density $f(x_1, \cdots, x_k)$ is the product of the individual marginal densities of $x_1, \cdots, x_k$ then the real scalar random variables $x_1, \cdots, x_k$ are said to be *mutually independently distributed*. If this property holds in two subsets of the variables then these subsets are said to be independently distributed. In such a case the joint density of the two subsets is the product of the marginal densities of the subsets and the conditional density of one subset given the other is free of the conditions or it is the marginal density of the first subset itself. Note the following:

Joint density = (conditional density) × (marginal density of the conditioned variables)

$$f(x_1, \cdots, x_k) = g(x_1, \cdots, x_r|x_{r+1}, \cdots, x_k) f_2(x_{r+1}, \cdots, x_k).$$

This is a general result.

$$g(x_1, \cdots, x_r|x_{r+1}, \cdots, x_k) = f_1(x_1, \cdots, x_r) = \text{ marginal density of } \{x_1, \cdots, x_r\}$$

if the two sets $\{x_1, \cdots, x_r\}$ and $\{x_{r+1}, \cdots, x_k\}$ are independently distributed . When $\{x_1, \cdots, x_r\}$ and $\{x_{r+1}, \cdots, x_k\}$ are independently distributed then

$$f(x_1,\cdots,x_k) = f_1(x_1,\cdots,x_r)f_2(x_{r+1},\cdots,x_k). \tag{1.3.5}$$

If $x_1,\cdots,x_k$ are mutually independently distributed then

$$f(x_1,\cdots,x_k) = \prod_{j=i}^{k} f_j(x_j) \tag{1.3.6}$$

where $f_j(x_j)$ is the marginal density x_j, $j = 1,\cdots,k$. Observe that independence in subsets need not imply mutual independence. For example x_1 and x_2 independent, x_1 and x_3 independent, x_2 and x_3 independent, need not imply that x_1, x_2, x_3 are mutually independent.

Definition 1.3.3. Joint moments or product moments.

$$M_{x_1,\cdots,x_k}(h_1,\cdots,h_k) = \int_{x_1}\cdots\int_{x_k} x_1^{h_1}\cdots x_k^{h_k} f(x_1,\cdots,x_k)\mathrm{d}x_1\wedge\cdots\wedge \mathrm{d}x_k \tag{1.3.7}$$

if it exists, where $f(x_1,\cdots,x_k)$ is the joint density of $x_1,\cdots,x_k$, is known as the expected value of the product $x_1^{h_1}\cdots x_k^{h_k}$, written as $E[x_1^{h_1}\cdots x_k^{h_k}]$, or the $(h_1,\cdots,h_k)^{th}$ product moment of $x_1,\cdots,x_k$. When $h_j = s_j - 1, j = 1,\cdots,k$ where $s_1,\cdots,s_k$ are arbitrary parameters, the product moment is known as the *Mellin transform of* $f(x_1,\cdots,x_k)$ when $x \geq 0, j = 1,\cdots,k$.

Definition 1.3.4. Moment generating functions.

Let $f(x_1,\cdots,x_k)$ be the joint density of the real scalar random variables $x_1,\cdots,x_k$ then the expected value of $\mathrm{e}^{(t_1x_1+\cdots+t_kx_k)}$, where $t_1,\cdots,t_k$, are arbitrary parameters, is known as the joint moment generating function of $x_1,\cdots,x_k$ when the expected value exists. When t_j is replaced by $-t_j$, that is $E[\mathrm{e}^{-t_1x_1-\cdots-t_kx_k}]$ is the *Laplace transform* of $f(x_1,\cdots,x_k)$ when $x_j \geq 0, j = 1,\cdots,k$. When t_j is replaced by it_j, $i = \sqrt{-1}, j = 1,\cdots,k$ then the expected value is the *Fourier transform* of the density $f(x_1,\cdots,x_k)$ or the *Characteristic function* of $x_1,\cdots,x_k$.

Moment generating function: $M_f(t_1,\cdots,t_k)$

$$\begin{aligned} M_f(t_1,\cdots,t_k) &= E[\mathrm{e}^{(t_1x_1+\cdots+t_kx_k)}] \\ &= \int_{x_1}\cdots\int_{x_k} \mathrm{e}^{t_1x_1+\cdots+t_kx_k} f(x_1,\cdots,x_k)\mathrm{d}x_1\wedge\cdots\wedge \mathrm{d}x_k. \end{aligned} \tag{1.3.8}$$

Characteristic function = Fourier transform of f or $\phi_f(t_1,\cdots,t_k)$:

$$\phi_f(t_1,\cdots,t_k) = E[\mathrm{e}^{(it_1x_1+it_2x_2+\cdots+it_kx_k)}],\ i=\sqrt{-1} \tag{1.3.9}$$
$$= \int_{x_1}\cdots\int_{x_k} \mathrm{e}^{it_1x_1+\cdots+it_kx_k} f(x_1,\cdots,x_k)\mathrm{d}x_1\wedge\cdots\wedge \mathrm{d}x_k.$$

Laplace transform of f or $L_f(t_1,\cdots,t_k)$:

$$L_f(t_1,\cdots,t_k) = E[\mathrm{e}^{(-t_1x_1-\cdots-t_kx_k)}], \tag{1.3.10}$$

when $x_1,\cdots,x_k$ are positive variables. That is,

$$L_f(t_1,\cdots,t_k) = M_f(-t_1,\cdots,-t_k)$$

when the variables are positive or

$$L_f(t_1,\cdots,t_k) = \int_0^\infty\cdots\int_0^\infty \mathrm{e}^{(-t_1x_1-\cdots-t_kx_k)} f(x_1,\cdots,x_k)\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_k \tag{1.3.11}$$

whenever $L_f(t_1,\cdots,t_k)$ exists. Note that $\phi_f(t_1,\cdots,t_k)$ exists always.

Exercises 1.3.

1.3.1. Check whether the following are density functions. If so, evaluate c.

(1) $f_1(x) = c\ \mathrm{e}^{-\theta x}$; (2) $f_2(x) = c\ \mathrm{e}^{5k}, 0\le x<\infty$; (3) $f_3(x) = c\ \mathrm{e}^{-\theta|x|}, -\infty<x<\infty$;

(4) $f_4(x) = c\mathrm{e}^{-\frac{1}{2\sigma^2}(x-\mu)^2}, -\infty<x,\infty,-\infty<\mu,\infty,\sigma>0$;

(5) $f_5(x) = \frac{\lambda_1}{\sqrt{2\pi}\sigma_1}\ \mathrm{e}^{-\frac{1}{2\sigma_1^2}(x-\mu_1)^2} + \frac{\lambda_2}{\sqrt{2\pi}\sigma_2}\ \mathrm{e}^{-\frac{1}{2\sigma_2^2}(x-\mu_2)^2}, -\infty<x<\infty,\ \lambda_1>0,$

$\lambda_2>0, \lambda_1+\lambda_2=1, \sigma_1>0, \sigma_2>0, -\infty<\mu_1<\infty, -\infty<\mu_2<\infty.$

1.3.2. Let $f(x)$ be a density function. Consider the distribution function or the cumulative density function of x, namely $F(x)=\int_{-\infty}^x f(t)\mathrm{d}t$. Let $y=F(x)$. What is the density of y?

1.3.3. Is the following a density function? If so, evaluate the marginal densities. $f(x_1,x_2) = c(x_1+x_2), 0\le x_1\le 1, 0\le x_2\le 1$, and $f(x_1,x_2)=0$ elsewhere.

1.3.4. Is the following a density function? If so, evaluate the probability that x falls in the interval $[1.5, 2.5]$, that is, $Pr\{1.5 \le x \le 2.5\}$ which is the integral over this interval.

$$f(x) = \begin{cases} cx, 0 \le x \le 2 & \text{and } f(x) = 0 \text{ elsewhere.} \\ (3-x), 2 \le x \le 3. \end{cases}$$

1.3.5. Check whether x_1 and x_2 in Exercise 1.3.3 are independently distributed.

1.3.6. In the joint density $f(x_1, x_2) = 2, 0 \le x_1 \le x_2 \le 1$,and $f(x_1, x_2) = 0$ elsewhere, evaluate (1) the conditional density of x_2 given $x_1 = \frac{1}{4}$; (2) the conditional density of x_1 given $x_2 = 2$.

1.3.7. Evaluate the product moment $E(x_1^2 x_2)$ in Exercise 1.3.6.

1.3.8. For the problem (1) of Exercise 1.3.1 evaluate the moment generating function. Then by using this moment generating function obtain the 4^{th} moment $E(x^4)$ by (1): expanding the moment generating function; (2): by differentiating the moment generating function.

1.3.9. For the density in Exercise 1.3.3 evaluate the joint moment generating function and show that

$$E[\mathrm{e}^{t_1x_1+t_2x_2}] = \frac{(\mathrm{e}^{t_2}-1)}{t_2}\Big[\frac{\mathrm{e}^{t_1}}{t_1} - \frac{(\mathrm{e}^{t_1}-1)}{t_1^2}\Big] + \Big(\frac{(\mathrm{e}^{t_1}-1)}{t_1}\Big)\Big[\frac{\mathrm{e}^{t_2}}{t_2} - \frac{(\mathrm{e}^{t_2}-1)}{t_2^2}\Big]; t_1 \ne 0, t_2 \ne 0.$$

1.3.10. For the normal density in (4) of Exercise 1.3.1 evaluate the *mean value* $= E(x)$ and the *variance* $= E[x - E(x)]^2$ and show that the mean value $E(x) = \mu$ and the variance $= \sigma^2$ there.

1.4. Gamma, Beta and Related Densities

A gamma density is associated with a gamma function. A two parameter gamma density is the following:

$$f(x) = \begin{cases} \frac{x^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha)}\mathrm{e}^{-x/\beta}; & x \ge 0, \beta > 0, \alpha > 0 \\ 0, & \text{elsewhere.} \end{cases} \tag{1.4.1}$$

For the gamma function to exist α can be complex also with the condition $\Re(\alpha) > 0$ where $\Re(.)$ denotes the real part of (.).

Example 1.4.1. Evaluate the following for a gamma density: (a) The h^{th} moment of the gamma random variable x; (b) the moment generating function of x; (c) the Laplace transform of f; (d) the Fourier transform of f or the characteristic function of x.

Solution: (a) The h-th moment is $E(x^h)$.

$$\begin{aligned} E(x^h) &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^h x^{\alpha-1} \mathrm{e}^{-x/\beta} \mathrm{d}x \qquad (\text{Put } y = \frac{x}{\beta}) \\ &= \beta^h \frac{\Gamma(\alpha+h)}{\Gamma(\alpha)} \quad \text{for } \Re(\alpha+h) > 0. \end{aligned} \tag{1.4.2}$$

Note that h can be negative also provided $\alpha + h > 0$ when α and h are real.

(b) Moment generating function of x, $M_x(t)$:

$$\begin{aligned} M_x(t) = E[\mathrm{e}^{\mathrm{tx}}] &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \mathrm{x}^{\alpha-1} \mathrm{e}^{\mathrm{tx}} \mathrm{e}^{-\mathrm{x}/\beta} \mathrm{dx} \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1} \mathrm{e}^{-(1-\beta t)x/\beta} \mathrm{d}x < \infty \text{ for } 1-\beta t > 0 \\ &= (1-\beta t)^{-\alpha} \text{ for } 1-\beta t > 0. \end{aligned} \tag{1.4.3}$$

Hence the Laplace transform of the gamma density is given by

$$L_f(t) = E[\mathrm{e}^{-\mathrm{tx}}] = (1+\beta \mathrm{t})^{-\alpha}, 1+\beta \mathrm{t} > 0. \tag{1.4.4}$$

The characteristic function of x or the Fourier transform of f:

$$\phi(t) = E[\mathrm{e}^{\mathrm{itx}}] = (1-\mathrm{i}\beta \mathrm{t})^{-\alpha}, \Re(1-\mathrm{i}\beta \mathrm{t}) > 0, \mathrm{i} = \sqrt{-1}. \tag{1.4.5}$$

Through the uniqueness of the Mellin and inverse Mellin transform pair, Laplace and inverse Laplace transform pair, Fourier and inverse Fourier transform pair, the density f is uniquely determined by $h = (s-1)^{th}$ moment in (1.4.2), the Laplace transform in (1.4.4) and the Fourier transform in (1.4.5).

1.4.1. The beta function and the beta density

Notation 1.4.1. $B(\alpha,\beta)$: **The beta function with parameters α and β**

Definition 1.4.1.

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \Re(\alpha)>0, \Re(\beta)>0. \tag{1.4.6}$$

Example 1.4.2. Derive integral representations for $B(\alpha,\beta)$ by using the integral representations of $\Gamma(\alpha)$ and $\Gamma(\beta)$.

Solution: From the integral representation of a gamma function in (1.1.13)

$$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x, \Re(\alpha)>0$$

and

$$\Gamma(\beta)=\int_0^\infty y^{\beta-1}\mathrm{e}^{-y}\mathrm{d}y, \Re(\beta)>0.$$

Hence,

$$\Gamma(\alpha)\Gamma(\beta)=\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}\mathrm{e}^{-(x+y)}\mathrm{d}x\wedge\mathrm{d}y.$$

Put $x=r\cos^2\theta, y=r\sin^2\theta, 0\le r<\infty, 0\le\theta\le\pi/2$. Then

$$\mathrm{d}x\wedge\mathrm{d}y=2r\cos\theta\sin\theta\ \ \mathrm{d}r\wedge\mathrm{d}\theta, x+y=r.$$

Integrating out r by using a gamma integral we obtain

$$\Gamma(\alpha)\Gamma(\beta)=\Gamma(\alpha+\beta)\int_{\theta=0}^{\pi/2}(\cos^2\theta)^{\alpha-1}(\sin^2\theta)^{\beta-1}2\cos\theta\sin\theta\mathrm{d}\theta.$$

Hence

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}=\int_{\theta=0}^{\pi/2}(\cos^2\theta)^{\alpha-1}(\sin^2\theta)^{\beta-1}2\cos\theta\sin\theta\mathrm{d}\theta. \tag{1.4.7}$$

Here (1.4.7) is one integral representation. Some others are the following, the necessary transformation is written in brackets.

$$
\begin{aligned}
B(\alpha,\beta) &= \int_{\theta=0}^{\pi/2} (\cos^2\theta)^{\alpha-1}(\sin^2\theta)^{\beta-1} 2\cos\theta\sin\theta\,\mathrm{d}\theta \\
&= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x, (x=\cos^2\theta) && (1.4.8)\\
&= \int_0^1 y^{\beta-1}(1-y)^{\alpha-1}\mathrm{d}y, (y=1-x) && (1.4.9)\\
&= \int_0^\infty u^{\alpha-1}(1+u)^{-(\alpha+\beta)}\mathrm{d}u, (u=\frac{x}{1-x}) && (1.4.10)\\
&= \int_0^\infty v^{\beta-1}(1+v)^{-(\alpha+\beta)}\mathrm{d}v, (v=\frac{1}{u}). && (1.4.11)
\end{aligned}
$$

Here (1.4.8) and (1.4.9) are known as the *type-1 integral representation of the beta function* $B(\alpha,\beta)$ and (1.4.10) and (1.4.11) are known as the *type-2 representation of the beta function*. Note also that $B(\alpha,\beta) = B(\beta,\alpha)$. Based on these representations we have the type-1 and type-2 beta densities.

Definition 1.4.2. The real type-1 beta density.

$$
f_1(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, 0 \le x \le 1, \Re(\alpha) > 0, \Re(\beta) > 0 \\ 0,\ \text{elsewhere}. \end{cases} \tag{1.4.12}
$$

Example 1.4.3. Evaluate the h^{th} moment of the real type-1 beta random variable x and the Mellin transform of the density $f_1(x)$.

Solution: The h-th moment is $E(x^h)$.

$$
\begin{aligned}
E(x^h) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{(\alpha+h)-1}(1-x)^{\beta-1}\mathrm{d}x \\
&= \frac{\Gamma(\alpha+h)}{\Gamma(\alpha)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+h)}, \Re(\alpha+h) > 0, && (1.4.13)
\end{aligned}
$$

evaluated from the normalizing constant $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$ by observing that the only change is that α is replaced by $\alpha+h$. By replacing h by $s-1$ we get the Mellin transform of the density f_1. That is,

$$M_{f_1}(s) = E(x^{s-1}) = \frac{\Gamma(\alpha+s-1)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+s-1)}, \Re(\alpha+s-1) > 0. \quad (1.4.14)$$

Definition 1.4.3. The real type-2 beta density

$$f_2(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1+x)^{-(\alpha+\beta)}, 0 \le x < \infty, \Re(\alpha) > 0, \Re(\beta) > 0 \\ 0, \text{ elsewhere.} \end{cases} \quad (1.4.15)$$

This is based on the type-2 integral representation of a beta function.

Example 1.4.4. Evaluate the h-th moment of a real type-2 beta random variable with the density f_2 above.

Solution:

$$\begin{aligned} E(x^h) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty x^{\alpha+h-1}(1+x)^{-(\alpha+\beta)}\mathrm{d}x \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty x^{(\alpha+h)-1}(1+x)^{-[(\alpha+h)+(\beta-h)]}\mathrm{d}x \\ &= \frac{\Gamma(\alpha+h)}{\Gamma(\alpha)}\frac{\Gamma(\beta-h)}{\Gamma(\beta)} \text{ for } \Re(\alpha+h) > 0, \Re(\beta-h) > 0 \end{aligned} \quad (1.4.16)$$

that is, $-\Re(\alpha) < \Re(h) < \Re(\beta)$. Note that only a few moments will exist here. When α, β, h are real then h has to be between $-\alpha$ and β.

Example 1.4.5. Let x and y be independently distributed real gamma random variables having the density as in (1.4.1) with the parameters $(\alpha_1, 1)$ and $(\alpha_2, 1)$ respectively. Let $u = x + y, v = \frac{x}{x+y}$ and $w = \frac{x}{y}$. Evaluate the densities of $u, v,$ and w.

Solution: Since the variables x and y are independently distributed the joint density of x and y is the product of the marginal densities of x and y. The marginal densities are given as gamma densities with parameters $(\alpha_1, 1)$ and $(\alpha_2, 1)$ respectively. Hence the joint density, denoted by $f(x, y)$, is the following:

$$f(x,y) = \begin{cases} \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha_1-1}y^{\alpha_2-1}, \mathrm{e}^{-(x+y)}, 0 \le x < \infty, 0 \le y < \infty. \\ 0, \text{elsewhere.} \end{cases} \quad (1.4.17)$$

Let us make the one-to-one transformation $x = r\cos^2\theta, y = r\sin^2\theta, 0 \le r < \infty, 0 \le \theta \le \pi/2$, with the Jacobian $\mathrm{d}x \wedge \mathrm{d}y = 2r\cos\theta\sin\theta\ \mathrm{d}r\mathrm{d}\theta$, then we obtain the joint density of r and θ denoted by $g(r,\theta)$. That is,

$$g(r,\theta) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}(r\cos^2\theta)^{\alpha_1-1}(r\sin^2\theta)^{\alpha_2-1}\mathrm{e}^{-r}(2r\cos\theta\sin\theta)$$

for $0 \le r < \infty, 0 \le \theta \le \pi/2$, and $g(r,\theta) = 0$ elsewhere. Since the variables are separated, we note that r and θ are independently distributed. Further, the marginal densities of r and θ, denoted by $g_1(r)$ and $g_2(\theta)$, are given by the following:

$$g_1(r) = c_1 r^{\alpha_1+\alpha_2-1}\mathrm{e}^{-r}, 0 \le r < \infty$$

and zero elsewhere and

$$g_2(\theta) = c_2(\cos^2\theta)^{\alpha_1-1}(\sin^2\theta)^{\alpha_2-1}(2\cos\theta\sin\theta), 0 \le \theta \le \pi/2$$

and zero elsewhere, where c_1 and c_2 are the normalizing constants. Since

$$\int_0^\infty g_1(r)\mathrm{d}r = 1 \Rightarrow c_1 = \frac{1}{\Gamma(\alpha_1+\alpha_2)}$$

and

$$\int_0^{\pi/2} g_2(\theta)\mathrm{d}\theta = 1 \Rightarrow c_2 = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}.$$

Hence the nonzero part of the densities are the following:

$$g_1(r) = \frac{r^{\alpha_1+\alpha_2-1}\mathrm{e}^{-r}}{\Gamma(\alpha_1+\alpha_2)}, 0 \le r < \infty, \Re(\alpha_1+\alpha_2) > 0 \tag{1.4.18}$$

and

$$g_2(\theta) = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}(\cos^2\theta)^{\alpha_1-1}(\sin^2\theta)^{\alpha_2-1}(2\cos\theta\sin\theta), \tag{1.4.19}$$

$0 \le \theta \le \pi/2$. Thus we have the following:

$$u = x + y = r\cos^2\theta + r\sin^2\theta = r$$

has a gamma density as in (1.4.18) with parameter $(\alpha_1+\alpha_2)$

$$v = \frac{x}{x+y} = \frac{r\cos^2\theta}{r\cos^2\theta + r\sin^2\theta} = \cos^2\theta.$$

Put $v = \cos^2\theta$ in (1.4.19) to obtain the density of v, denoted by $g_3(v)$, as follows:

$$g_3(v) = \begin{cases} \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} v^{\alpha_1-1}(1-v)^{\alpha_2-1}, 0 \le v \le 1, \Re(\alpha_1) > 0,\ \Re(\alpha_2) > 0, \\ 0,\ \text{elsewhere.} \end{cases} \tag{1.4.20}$$

Hence v has a real type-1 beta distribution with the parameters (α_1, α_2) as in (1.4.20)

$$w = \frac{x}{y} = \frac{r\cos^2\theta}{r\sin^2\theta} = \cot^2\theta \Rightarrow \frac{1}{1+w} = \sin^2\theta \Rightarrow$$

$$2\sin\theta\cos\theta\,\mathrm{d}\theta = \frac{1}{(1+w)^2}\,\mathrm{d}w \text{ and } \cos^2\theta = 1-\sin^2\theta = 1-\frac{1}{1+w} = \frac{w}{1+w}.$$

Therefore,

$$g_2(\theta)\left|\frac{\mathrm{d}\theta}{\mathrm{d}w}\right| = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\Big(\frac{w}{1+w}\Big)^{\alpha_1-1}\Big(\frac{1}{1+w}\Big)^{\alpha_2-1}\frac{1}{(1+w)^2} \Rightarrow$$

$$g_4(w) = \begin{cases} \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} w^{\alpha_1-1}(1+w)^{-(\alpha_1+\alpha_2)}\ , 0 \le w < \infty \\ 0,\ \text{elsewhere.} \end{cases} \tag{1.4.21}$$

Hence $w = \frac{x}{y}$ has a real type-2 beta distribution as given in (1.4.21). Thus $u = x_1 + x_2 \sim$ gamma $(\alpha_1 + \alpha_2, 1), v = \frac{x}{x+y} \sim$ type-1 beta $(\alpha_1, \alpha_2), w = \frac{x}{y} \sim$ type-2 beta (α_1, α_2), where "$\sim$" indicates "distributed as " and the parameters are given in the bracket.

Exercises 1.4.

1.4.1. If $x_1, \cdots, x_k$ are mutually independently distributed real scalar random variables and if $\phi_1(x_1), \phi_2(x_2), \cdots, \phi_k(x_k)$ are functions of $x_1, \cdots, x_k$ respectively, then show that the expected value of a product is the product of the expectations that is,

$$E[\phi_1(x_1)\cdots\phi_k(x_k)] = E[\phi_1(x_1)]E[\phi_2(x_2)]\cdots E[\phi_k(x_k)].$$

1.4.2. If x is a real scalar random variable with density $f(x)$ and if a and b are constant scalars then show that (1) $E(b) = b$; (2) $E[a\phi(x) + b] = aE[\phi(x)] + b$ where $\phi(x)$ is a function of x.

1.4.3. Show that (1) $\mathrm{Var}[ax + b] = a^2\mathrm{Var}(x)$ where a and b are constants and $\mathrm{Var}(x)$ is the variance of x.

1.4.4. Show that $\mathrm{Var}(x) = E(x^2) - [E(x)]^2$, for any real scalar random variable x.

1.4.5. The covariance between two real scalar random variables x and y is denoted and defined as $\mathrm{Cov}(x_1, x_2) = E[x - E(x)][y - E(y)]$. Show that $\mathrm{Cov}(x, y) = E(xy) - E(x)E(y)$.

1.4.6. If x has the uniform density $f(x) = \frac{1}{\theta}, 0 \le x \le \theta$ and zero elsewhere, evaluate the variance of $2x + 5$.

1.4.7. From the joint density, $f(x, y) = 2, 0 \le x \le y \le 1$ and $f(x, y) = 0$ elsewhere, evaluate $\mathrm{Cov}(x, y)$.

1.4.8. Linear correlation coefficient ρ between real scalar random variables x and y, is defined as $\rho = \frac{\mathrm{Cov}(x,y)}{\sqrt{\mathrm{Var}(x)\mathrm{Var}(y)}}$ for $\mathrm{Var}(x) \neq 0, \mathrm{Var}(y) \neq 0$. Show that $-1 \le \rho \le 1$ and that $\rho = \pm 1$ if and only if x and y are linearly related.

1.4.9. Let $x_1, \cdots, x_k$ be independently distributed real type-1 beta random variables with the parameters $(\alpha_1, \beta_1) \cdots (\alpha_k, \beta_k)$. Let $u = x_1 x_2 \cdots x_k$ the product of these variables. Evaluate the h-th moment of u.

1.4.10. Let x_1 be type-1 beta with parameters (α_1, β_1), x_2 be type-2 beta with parameters (α_2, β_2) and x_3 be a gamma random variable with the parameters $(\alpha_3, 1)$. Let x_1, x_2, x_3 be mutually independently distributed. Let $u = \frac{x_1 x_2}{x_3}$. Evaluate the h-th moment of u and write down the conditions for its existence.

1.5. Hypergeometric Series

A general hypergeometric series with p upper or numerator parameters and q lower or denominator parameters is denoted and defined as follows:

Notation 1.5.1.

$$ {}_pF_q\,(a_1 \cdots a_p; b_1 \cdots b_q; z) = {}_pF_q((a_p); (b_q); z) = {}_pF_q(z) $$

Definition 1.5.1.

$$ {}_pF_q(z) = \sum_{r=0}^{\infty} \frac{(a_1)_r \cdots (a_p)_r}{(b_1)_r \cdots (b_q)_r} \frac{z^r}{r!} \tag{1.5.1} $$

where $(a_j)_r$ and $(b_j)_r$ are the Pochammer symbols of (1.1.1). The series in (1.5.1) is defined when none of the b_j 's, $j = 1, \cdots q$, is a negative integer or zero. If a b_j is a negative integer or zero then $(b_j)_r$ will be zero for some r. A b_j can be zero provided there is a numerator parameter a_k such that $(a_k)_r$ becomes zero first before

$(b_j)_r$ becomes zero. If any numerator parameter a_j is a negative integer or zero then (1.5.1) terminates and becomes a polynomial in z. From the ratio test it is evident that the series in (1.5.1) is convergent for all z if $q \geq p$, it is convergent for $|z| < 1$ if $p = q + 1$ and divergent if $p > q + 1$. When $p = q + 1$ and $|z| = 1$ the series can converge in some cases. Let

$$\beta = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j.$$

It can be shown that when $p = q + 1$ the series is absolutely convergent for $|z| = 1$ if $\Re(\beta) < 0$, conditionally convergent for $z = -1$ if $0 \leq \Re(\beta) < 1$ and divergent for $|z| = 1$ if $1 \leq \Re(\beta)$.

Some special cases of a ${}_pF_q$ are the following: When there is no upper or lower parameters we have,

$$ {}_0F_0(;;\pm z) = \sum_{r=0}^{\infty} \frac{(\pm z)^r}{r!} = \mathrm{e}^{\pm z}. \tag{1.5.2}$$

Thus ${}_0F_0(.)$ is an exponential series.

$$ {}_1F_0(\alpha;;z) = \sum_{r=0}^{\infty} (\alpha)_r \frac{z^r}{r!} = (1-z)^{-\alpha} \text{ for } |z| < 1. \tag{1.5.3}$$

This is the binomial series. ${}_1F_1(.)$ is known as confluent hypergeometric series or *Kummers's hypergeometric series and* ${}_2F_1(.)$ is known as *Gauss' hypergeometric series* .

Example 1.5.1. **Incomplete Gamma function**. Evaluate the incomplete gamma function

$$\gamma(\alpha, b) = \int_0^b x^{\alpha-1} \mathrm{e}^{-x} \mathrm{d}x, \; b < \infty$$

and write it in terms of a Kummer's hypergeometric function.

Solution: Since b is finite we may expand the exponential part and integrate term by term .

$$\begin{aligned}\gamma(\alpha, b) &= \int_0^b x^{\alpha-1}\left\{\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}x^r\right\}\mathrm{d}x = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\int_0^b x^{\alpha+r-1}\mathrm{d}x \\ &= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{b^{\alpha+r}}{\alpha+r} = \frac{b^\alpha}{\alpha}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{(\alpha)_r}{(\alpha+1)_r}b^r \\ &= \frac{b^\alpha}{\alpha}\,{}_1F_1(\alpha;\alpha+1;-b). \end{aligned} \tag{1.5.4}$$

Hence the upper part

$$\Gamma(\alpha, b) = \int_b^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x = \Gamma(\alpha) - \gamma(\alpha, b). \tag{1.5.5}$$

Example 1.5.2. **Incomplete beta function**. Evaluate the incomplete beta function

$$b(\alpha,\beta;t) = \int_0^t x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x, t < 1$$

and write it in terms of a Gauss' hypergeometric function.

Solution: Note that since $0 < x < 1$,

$$(1-x)^{\beta-1} = (1-x)^{-(1-\beta)} = \sum_{r=0}^{\infty}\frac{(1-\beta)_r}{r!}x^r.$$

Hence,

$$\begin{aligned}b(\alpha,\beta;t) &= \sum_{r=0}^{\infty}\frac{(1-\beta)_r}{r!}\int_0^t x^{\alpha+r-1}\mathrm{d}x = \sum_{r=0}^{\infty}\frac{(1-\beta)_r}{r!}\frac{t^{\alpha+r}}{\alpha+r} \\ &= \frac{t^\alpha}{\alpha}\sum_{r=0}^{\infty}\frac{(1-\beta)_r(\alpha)_r}{(\alpha+1)_r}\frac{t^r}{r!} = \frac{t^\alpha}{\alpha}\,{}_2F_1(1-\beta,\alpha;\alpha+1;t). \end{aligned} \tag{1.5.6}$$

Hence the upper part,

$$B(\alpha,\beta;t) = \int_t^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} - b(\alpha,\beta;t). \tag{1.5.7}$$

Example 1.5.3. Obtain an integral representation for a $_2F_1$.

Solution: Consider the integral,

$$\int_0^1 x^{a-1}(1-x)^{c-a-1}(1-zx)^{-b}\mathrm{d}x, \text{for } |z|<1$$
$$= \sum_{r=0}^{\infty} \frac{(z)^r}{r!}(b)_r \int_0^1 x^{a+r-1}(1-x)^{c-a-1}\mathrm{d}x, \text{(expanding } (1-zx)^{-b}$$

by binomial expansion)

$$= \sum_{r=0}^{\infty} \frac{(z)^r}{r!}(b)_r \frac{\Gamma(a+r)\Gamma(c-a)}{\Gamma(c+r)} \text{ (by using a type-1 beta integral)}$$
$$= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} \sum_{r=0}^{\infty} \frac{(z)^r}{r!} \frac{(b)_r(a)_r}{(c)_r} \text{ (by writing } \Gamma(a+r)=(a)_r\Gamma(a))$$
$$= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} {_2F_1}(a,b;c;z).$$

That is,

$$_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 x^{a-1}(1-x)^{c-a-1}(1-zx)^{-b}\mathrm{d}x \qquad (1.5.8)$$
$$\text{for } \Re(a)>0,\ \Re(c-a)>0.$$

This is the famous integral representation for $_2F_1$.

1.5.1. Evaluation of some contour integrals

Since the technique of Mellin and inverse Mellin transforms is frequently used for solving some problems in applied areas we will look into the evaluation of some contour integrals with the help of residue theorem. We will not go into the theory of analytic functions and residue calculus. We will need to know only how to apply the residue theorem for evaluating some integrands where the integrands contain gamma functions. In order to illustrate the technique let us redo a known result.

Example 1.5.4. Evaluate the Mellin transform of e^{-x} and then recover $\mathrm{e}^{-x}, 0 \le x < \infty$ with the help of the inverse Mellin transform.

Solution: Let $f = e^{-x}$ and $M_f(s)$ its Mellin transform. Then

$$M_f(s) = \int_0^{\infty} x^{s-1} e^{-x} dx = \Gamma(s) \text{ for } \Re(s) > 0. \tag{1.5.9}$$

The inverse Mellin transform of $\Gamma(s)$ is given by the formula

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) x^{-s} ds, i = \sqrt{-1}, c > 0.$$

Then from residue calculus $f(x)$ is available as the sum of residues at the poles of the integrand $\Gamma(s)x^{-s}$. The poles are coming from $\Gamma(s)$, which are at the points $s = -\nu, \nu = 0, 1, 2 \cdots$. The residue at $s = -\nu$ is given by the following, denoted by $\Re_\nu$:

$$\Re_\nu = \lim_{s \to -\nu} [(s + \nu)\Gamma(s)x^{-s}].$$

Direct substitution of $s = -\nu$ fails here. Hence we may adopt the following procedure.

$$\begin{aligned} \Re_\nu &= \lim_{s \to -\nu} [(s + \nu) \frac{(s + \nu - 1) \cdots s\Gamma(s)}{(s + \nu - 1) \cdots s} x^{-s}] \\ &= \lim_{s \to -\nu} [\frac{\Gamma(s + \nu + 1)x^{-s}}{(s + \nu - 1) \cdots s}] = \frac{\Gamma(1)x^\nu}{(-1)(-2) \cdots (-\nu)} = \frac{(-1)^\nu}{\nu!} x^\nu. \end{aligned}$$

Hence the sum of the residues should produce $f(x)$.

$$f(x) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} x^\nu = e^{-x}.$$

Example 1.5.5. Evaluate the contour integral, which is also an inverse Mellin transform,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(a_1 - s) \cdots \Gamma(a_p - s)}{\Gamma(b_1 - s) \cdots \Gamma(b_q - s)} \Gamma(s)(-z)^{-s} ds \tag{1.5.10}$$

as the sum of the residues at the pole of $\Gamma(s)$.

Solution: The poles are at $s = -\nu, \nu = 0, 1, \cdots$. The residue at $s = -\nu$ is given by the following:

$$\Re_\nu = \lim_{s \to -\nu} \left\{ (s + \nu)\Gamma(s) \frac{\Gamma(a_1 - s) \cdots \Gamma(a_p - s)}{\Gamma(b_1 - s) \cdots \Gamma(b_q - s)} (-z)^{-s} \right\}.$$

By using the process in Example 1.5.4 we have,

$$\begin{aligned}\mathfrak{R}_\nu &= \frac{(-1)^\nu}{\nu!}\frac{\Gamma(a_1+\nu)\cdots\Gamma(a_p+\nu)}{\Gamma(b_1+\nu)\cdots\Gamma(b_q+\nu)}(-z)^\nu\\ &= \left\{\frac{\prod_{j=1}^{p}\Gamma(a_j)}{\prod_{j=1}^{q}\Gamma(b_j)}\right\}\frac{(a_1)_\nu\cdots(a_p)_\nu}{(b_1)_\nu\cdots(b_q)_\nu}\frac{z^\nu}{\nu!}.\end{aligned}$$

Hence the sum of the residues is given by,

$$\sum_{\nu=0}^{\infty}\mathfrak{R}_\nu = K\sum_{\nu=0}^{\infty}\frac{(a_1)_\nu\cdots(a_p)_\nu}{(b_1)_\nu\cdots(b_q)_\nu}\frac{z^\nu}{\nu!} = K\ {}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;z)$$

where K is the constant

$$K = \frac{\prod_{j=1}^{p}\Gamma(a_j)}{\prod_{j=1}^{q}\Gamma(b_j)}.$$

Thus $\frac{1}{K}$ times the right side in (1.5.10) is the Mellin-Barnes representation for a general hypergeometric function.

If poles of higher orders are involved then one may use the general formula. If $\phi(z)$ has a pole of order m at $z = a$ then the residue at $z = a$, denoted by $\mathfrak{R}_a$, is given by the following formula:

$$\mathfrak{R}_a = \lim_{z\to a}\left\{\frac{1}{(m-1)!}[\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}(z-a)^m\phi(z)]\right\}. \qquad (1.5.11)$$

Some illustrations of this formula will be given when we solve some problems in astrophysics later on.

Exercises 1.5.

1.5.1. For a Gauss' hypergeometric function ${}_2F_1$ derive the following relationships:

$$\begin{aligned}{}_2F_1(a,b;c;z) &= (1-z)^{-b}\ {}_2F_1(c-a,b;c;-\frac{z}{1-z}), z\neq 1\\ &= (1-z)^{-a}\ {}_2F_1(a,c-b;c;-\frac{z}{1-z}), z\neq 1\\ &= (1-z)^{c-a-b}\ {}_2F_1(c-a,c-b;c;z).\end{aligned}$$

1.5.2. Show that

$$ {}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \Re(c-a)>0, \Re(c-a-b)>0, \Re(a)>0. $$

1.5.3. Let x_1 and x_2 be independently distributed real scalar gamma random variables with the parameters $(\alpha_1, 1)$ and $(\alpha_2, 1)$ respectively. Let $u = x_1 x_2$. Evaluate the density of u by using Mellin transformation technique when α_1 and α_2 do not differ by integers or zero.

1.5.4. Let x_1 and x_2 be independently distributed real type-1 beta random variables with the parameters (α_1, β_1) and (α_2, β_2) respectively. Let $u = x_1 x_2$. Evaluate the density of u by using Mellin transform technique if α_1 and α_2 do not differ by integers or zero.

1.5.5. Repeat the problem in Excercise 1.5.4 if x_1 and x_2 are type-2 beta distributed, where $\alpha_1 - \alpha_2 \neq \pm\lambda, \lambda = 0, 1, \cdots \beta_1 - \beta_2 \neq \pm\nu, \nu = 0, 1, 2, \cdots$.

1.5.6. Let $f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\alpha-s)\Gamma(s)x^{-s}\mathrm{d}s$. Evaluate $f(x)$ as the sum of residues at the poles of $\Gamma(s)$. Then evaluate it again at the poles of $\Gamma(\alpha - s)$. Then compare the two results. In the first case we get the function for $|x| < 1$ and in the case for $|x| > 1$.

1.5.7. Bose-Einstein density in physics: Let

$$ f(x)=\frac{1}{c[-1+\mathrm{e}^{\alpha+\beta x}]}, 0 \le x < \infty, \beta > 0. $$

Evaluate the normalizing constant c in this Bose-Einstein density.

1.5.8. Fermi-Dirac density in physics: Let

$$ f(x)=\frac{1}{c[1+\mathrm{e}^{\alpha+\beta x}]}, \beta > 0, 0 \le x < \infty. $$

Evaluate the normalizing constant c.

1.5.9. Maxwell-Boltzmann density in physics: Let

$$ f(x) = cx^2\mathrm{e}^{-\beta x^2}, 0 \le x < \infty, \beta > 0. $$

Evaluate c.

1.5.10. Generalized real gamma density. Let

$$ f(x) = cx^{\alpha-1}\mathrm{e}^{-bx^{\delta}}, b > 0, 0 \le x < \infty. $$

Evaluate the normalizing constant c.

1.6. Meijer's G-function

A generalization of the hypergeometric function in the real scalar case is Meijer's G-Function. It is defined in terms of a Mellin-Barnes integral.

Notation 1.6.1.

$$G_{p,q}^{m,n}[z|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}] = G_{p,q}^{m,n}[z|_{(b_q)}^{(a_p)}] = G_{p,q}^{m,n}(z) = G(z).$$

Definition 1.6.1. G-function.

$$G_{p,q}^{m,n}[z|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}] = \frac{1}{2\pi i}\int_L \frac{\left\{\prod_{j=1}^{m}\Gamma(b_j+s)\right\}\left\{\prod_{j=1}^{n}\Gamma(1-a_j-s)\right\}z^{-s}\mathrm{d}s}{\left\{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)\right\}\left\{\prod_{j=n+1}^{p}\Gamma(a_j+s)\right\}} \tag{1.6.1}$$

where L is a contour separating the poles of $\Gamma(b_j+s),\ j=1,\cdots,m$ from those of $\Gamma(1-a_j-s),\ j=1,\cdots,n$. Three types of contours are described and the conditions of existence for the G-function are discussed in Mathai (1993). The simplified conditions are the following: $G(z)$ exists for the following situations:

$$\begin{aligned} &\text{(i)} && q\geq 1, q>p, \text{ for all } z, z\neq 0\\ &\text{(ii)} && q\geq 1, q=p, \text{ for } |z|<1\\ &\text{(iii)} && p\geq 1, p>q, \text{ for all } z, z\neq 0\\ &\text{(iv)} && p\geq 1, p=q, \text{ for } |z|>1. \end{aligned} \tag{1.6.2}$$

Example 1.6.1. Evaluate

$$f(x) = G_{1,1}^{1,0}\left[x|_{\alpha}^{\alpha+\beta+1}\right].$$

Solution: As per our notation, $m=1, n=0, p=1, q=1$.

$$G_{1,1}^{1,0}\left[x|_{\alpha}^{\alpha+\beta+1}\right] = \frac{1}{2\pi i}\int_L \frac{\Gamma(\alpha+s)}{\Gamma(\alpha+\beta+1+s)}x^{-s}\mathrm{d}s.$$

As per situation (ii) above we should obtain a convergent function for $|x|<1$ if we evaluate the integral as the sum of the residues at the poles of $\Gamma(\alpha+s)$. The poles are at $s=-\alpha-\nu, \nu=0,1,\cdots$ and the sum of the residues

$$\sum_{\nu=0}^{\infty}\Re_\nu = \sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}\frac{x^{\nu+\alpha}}{\Gamma(\beta+1-\nu)}; \Gamma(\beta+1-\nu) = \frac{(-1)^\nu\Gamma(\beta+1)}{(-\beta)_\nu}.$$

$$G_{1,1}^{1,0}[x|_{\alpha}^{\alpha+\beta+1}] = \frac{x^\alpha}{\Gamma(\beta+1)}\sum_{\nu=0}^{\infty}\frac{(-\beta)_\nu x^\nu}{\nu!} = \frac{x^\alpha}{\Gamma(\beta+1)}(1-x)^\beta, |x|<1 \qquad (1.6.3)$$

for $\Re(\beta+1) > 0$.

Example 1.6.2. Let $u = x_1 x_2 \cdots x_p$ where $x_1, \cdots, x_p$ are independently distributed real random variables with (1) : x_j gamma distributed with parameters $(\alpha_j, 1)$, $j = 1, \cdots, p$; (2) : x_j type-1 beta distributed with parameters (α_j, β_j), $j = 1, \cdots, p$; (3) : x_j is type-2 beta distributed with parameters (α_j, β_j), $j = 1, \cdots, p$. Evaluate the density of u in (1),(2) and (3).

Solution: Taking the $(s-1)^{th}$ moment of u or the Mellin transform of the density of u we have the following:

$$E(u^{s-1}) = E(x_1\cdots x_p)^{s-1} = E(x_1^{s-1}\cdots x_p^{s-1}) = E(x_1^{s-1})\cdots E(x_p^{s-1})$$

due to independence

$$= \prod_{j=1}^{p} E(x_j^{s-1}) = \prod_{j=1}^{p}\frac{\Gamma(\alpha_j+s-1)}{\Gamma(\alpha_j)}, \Re(\alpha_j+s-1) > 0, j = 1, \cdots, p$$

for case (1)

$$= \prod_{j=1}^{p}\frac{\Gamma(\alpha_j+s-1)}{\Gamma(\alpha_j)}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j+\beta_j+s-1)}, \Re(\alpha_j+s-1) > 0, j = 1, \cdots, p$$

for case (2)

$$= \prod_{j=1}^{p}\frac{\Gamma(\alpha_j+s-1)}{\Gamma(\alpha_j)}\frac{\Gamma(\beta_j-s+1)}{\Gamma(\beta_j)}, \Re(\alpha_j+s-1) > 0, \Re(\beta_j-s+1) > 0,$$

$j = 1, \cdots, p$ for case (3).

Let the densities be denoted by $g_1(u), g_2(u)$ and $g_3(u)$ respectively. They are available from the respective inverse Mellin transforms which can be written as G-functions as follows:

$$g_1(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{\prod_{j=1}^{p}\frac{\Gamma(\alpha_j - 1 + s)}{\Gamma(\alpha_j)}\right\}u^{-s}\mathrm{d}s$$
$$= \frac{1}{\left\{\prod_{j=1}^{p}\Gamma(\alpha_j)\right\}}G_{0,p}^{p,0}[u\big|_{\alpha_j-1, j=1,\cdots,p}], \text{ for } u > 0, \Re(\alpha_j) > 0, j = 1, \cdots, p \tag{1.6.4}$$

and zero elsewhere.

$$g_2(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{\prod_{j=1}^{p}\frac{\Gamma(\alpha_j + s - 1)}{\Gamma(\alpha_j)}\frac{\Gamma(\alpha_j + \beta_j)}{\Gamma(\alpha_j + \beta_j + s - 1)}\right\}u^{-s}\mathrm{d}s$$
$$= \left\{\prod_{j=1}^{p}\frac{\Gamma(\alpha_j + \beta_j)}{\Gamma(\alpha_j)}\right\}G_{p,p}^{p,0}[u\big|_{\alpha_j-1, j=1,\cdots,p}^{\alpha_j+\beta_j-1, j=1,\cdots,p}], 0 < u < 1, \tag{1.6.5}$$
$$\Re(\alpha_j) > 0, \Re(\beta_j) > 0, \text{and zero elsewhere .}$$

$$g_3(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{\prod_{j=1}^{p}\frac{\Gamma(\alpha_j + s - 1)}{\Gamma(\alpha_j)}\frac{\Gamma(\beta_j - s + 1)}{\Gamma(\beta_j)}\right\}u^{-s}\mathrm{d}s \tag{1.6.6}$$
$$= \frac{1}{\left\{\prod_{j=1}^{p}\Gamma(\alpha_j)\Gamma(\beta_j)\right\}}G_{p,p}^{p,p}[u\big|_{\alpha_j-1, j=1,\cdots,p}^{-\beta_j, j=1,\cdots,p}], u > 0,$$
$$\Re(\alpha_j) > 0, \Re(\beta_j) > 0, j = 1, \cdots, p \text{ and zero elsewhere.}$$

Example 1.6.3. Evaluate the following integral, a particular case of which is the reaction rate integral in astrophysics.

$$I(\alpha, a, b, \rho) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-ax-bx^{-\rho}}\mathrm{d}x, a > 0, b > 0, \rho > 0. \tag{1.6.7}$$

Solution: Since the integrand can be taken as a product of positive integrable functions we can apply statistical distribution theory to evaluate this integral or such similar integrals. The procedure to be discussed here is suitable for a wide variety of problems. Let x_1 and x_2 be two real scalar random variables with density functions

$f_1(x_1)$ and $f_2(x_2)$. Let $u = x_1x_2$ and let x_1 and x_2 be independently distributed. Then the joint density of x_1 and x_2, denoted by $f(x_1, x_2)$, is the product of the marginal densities due to statistical independence of x_1 and x_2. That is,

$$f(x_1, x_2) = f_1(x_1)f_2(x_2).$$

Consider the transformation $u = x_1x_2$ and $v = x_1 \Rightarrow \mathrm{d}x_1 \wedge \mathrm{d}x_2 = \frac{1}{v}\mathrm{d}u \wedge \mathrm{d}v$. Hence the joint density of u and v, denoted by $g(u, v)$, is available as,

$$g(u, v) = \frac{1}{v} f_1(v) f_2(\frac{u}{v}). \tag{1.6.8}$$

Then the density of u denoted by $g_1(u)$, is available by integrating out v from $g(u, v)$. That is,

$$g_1(u) = \int_v \frac{1}{v} f_1(v) f_2(\frac{u}{v})\mathrm{d}v. \tag{1.6.9}$$

Here (1.6.8) and (1.6.9) are general results and the method described here is called the *method of transformation of variables* for obtaining the density of $u = x_1x_2$. Now, consider (1.6.7). Let

$$f_1(x_1) = c_1x_1^{\alpha}\mathrm{e}^{-ax_1} \text{ and } f_2(x_2) = c_2\mathrm{e}^{-zx_2^{\rho}}, 0 \le x_1 < \infty, 0 \le x_2 < \infty \tag{1.6.10}$$

$a > 0, z > 0$, where c_1 and c_2 are the normalizing constants. These normalizing constants can be evaluated by using the property.

$$1 = \int_0^{\infty} f_1(x_1)\mathrm{d}x_1 \text{ and } 1 = \int_0^{\infty} f_2(x_2)\mathrm{d}x_2.$$

Since we do not need the explicit forms of c_1 and c_2 we will not evaluate them here. With the f_1 and f_2 in (1.6.10) let us evaluate (1.6.9). We have

$$g_1(u) = c_1c_2 \int_{v=0}^{\infty} \frac{1}{v} v^{\alpha}\mathrm{e}^{-av}\mathrm{e}^{-z(\frac{u}{v})^{\rho}}\mathrm{d}v = c_1c_2 \int_{v=0}^{\infty} v^{\alpha-1}\mathrm{e}^{-av}\mathrm{e}^{-(zu^{\rho})v^{-\rho}}\mathrm{d}v. \tag{1.6.11}$$

Note that with $b = zu^{\rho}$, (1.6.11) is (1.6.7) multiplied by c_1 and c_2. Thus, we have identified the integral to be evaluated as a constant multiple of the density of u. This density of u is unique. Let us evaluate the density through Mellin and inverse Mellin transform technique.

$$E(u^{s-1}) = E(x_1^{s-1})E(x_2^{s-1})$$

due to statistical independence of x_1 and x_2. But

$$E(x_1^{s-1}) = c_1 \int_0^\infty x_1^{\alpha+s-1} e^{-ax_1} dx_1 = c_1 a^{-(\alpha+s)} \Gamma(\alpha+s), \Re(\alpha+s) > 0 \quad (1.6.12)$$

and

$$E(x_2^{s-1}) = c_2 \int_0^\infty x_2^{s-1} e^{-zx_2^{\rho}} dx_2 = \frac{c_2}{\rho z^{s/\rho}} \int_0^\infty y^{\frac{s}{\rho}-1} e^{-y} dy = \frac{c_2}{\rho z^{s/\rho}} \Gamma(\frac{s}{\rho}), \Re(s) > 0. \quad (1.6.13)$$

Hence

$$E(u^{s-1}) = c_1 c_2 \frac{a^{-\alpha}}{\rho} (az^{\frac{1}{\rho}})^{-s} \Gamma(\alpha+s) \Gamma(\frac{s}{\rho}), \quad (1.6.14)$$

Therefore, the density of u, denoted by $g_1(u)$, is available from the inverse Mellin transform.

$$g_1(u) = c_1 c_2 \frac{a^{-\alpha}}{\rho} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\alpha+s) \Gamma(\frac{s}{\rho}) (az^{\frac{1}{\rho}} u)^{-s} \, ds. \quad (1.6.15)$$

Now, compare (1.6.15) with (1.6.11) to obtain the following:

$$\int_0^\infty v^{\alpha-1} e^{-av} e^{-(zu^{\rho})v^{-\rho}} dv = \frac{a^{-\alpha}}{\rho} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\alpha+s) \Gamma(\frac{s}{\rho}) (az^{\frac{1}{\rho}} u)^{-s} \, ds. \quad (1.6.16)$$

On the right side in (1.6.15) the coefficient of s in $\Gamma(\frac{s}{\rho})$ is $\frac{1}{\rho} \neq 1$. Hence (1.6.15) is not a G-function but it can be written as an H-function, which will be considered next. In reaction rate theory in physics $\rho = \frac{1}{2}$ and then

$$\Gamma(\frac{s}{\rho}) = \Gamma(2s) = \pi^{\frac{1}{2}} 2^{2s-1} \Gamma(s) \Gamma\left(s + \frac{1}{2}\right)$$

by using the duplication formula for gamma functions. Then the right side of (1.6.16) reduces to

$$\begin{aligned} &\frac{1}{2\rho a^{\alpha} \pi^{\frac{1}{2}}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\alpha+s) \Gamma(s) \Gamma\left(s+\frac{1}{2}\right) \left(\frac{auz^{1/\rho}}{4}\right)^{-\rho} ds \\ &= \frac{1}{2\rho a^{\alpha} \pi^{\frac{1}{2}}} G_{0,3}^{3,0}\left[\frac{auz^{1/\rho}}{4}\Big|_{\alpha,0,\frac{1}{2}}\right], u > 0. \end{aligned}$$

But

$$b = zu^{\rho} \Rightarrow \frac{auz^{1/\rho}}{4} = \frac{ab^{1/\rho}}{4}.$$

Hence, for $\rho = \frac{1}{2}$,

$$\int_0^\infty v^{\alpha-1}\mathrm{e}^{-av-bv^{-\rho}}\mathrm{d}v = \frac{1}{2\rho a^\alpha \pi^{\frac{1}{2}}}G^{3,0}_{0,3}\Big[\frac{ab^{1/\rho}}{4}\Big|_{\alpha,0,\frac{1}{2}}\Big] \text{ for } \rho=\frac{1}{2}$$
$$= \frac{1}{a^\alpha \pi^{\frac{1}{2}}}G^{3,0}_{0,3}\Big[\frac{ab^2}{4}\Big|_{\alpha,0,\frac{1}{2}}\Big], u>0. \tag{1.6.17}$$

Exercises 1.6.

Write down the Mellin-Barnes representations in Excercises 1.6.1.- 1.6.5 where the series forms are given. Here is an illustration.

$$ {}_1F_0(\alpha;;x) = \sum_{r=0}^{\infty}(\alpha)_r\frac{x^r}{r!} = \frac{1}{\Gamma(\alpha)}\sum_{r=0}^{\infty}\Gamma(\alpha+r)\frac{x^r}{r!}$$
$$= \frac{1}{\Gamma(\alpha)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\alpha-s)\Gamma(s)(-x^{-s})\mathrm{d}s.$$

The last expression is the Mellin-Barnes representation for the series form ${}_1F_0(\alpha;;x)$.

1.6.1. ${}_0F_0(;;-z) = \mathrm{e}^{-z} = \sum_{r=0}^{\infty}\frac{(-z)^r}{r!}$ (Exponential Series)

1.6.2. ${}_2F_1(a,b;c;z) = \sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r}\frac{(z)^r}{r!}$ (Gauss' hypergeometric series)

1.6.3. ${}_1F_1(a;b;z) = \sum_{r=0}^{\infty}\frac{(a)_r}{(b)_r}\frac{(z)^r}{r!}$ (Confluent hypergeometric series)

1.6.4. $\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{(z/2)^{\nu+2r}}{\Gamma(\nu+r+1)}$ (Bessel function $J_\nu(z)$)

1.6.5. $\sum_{r=0}^{\infty}\frac{(z/2)^{\nu+2r}}{r!\Gamma(\nu+r+1)}$ (Bessel function $I_\nu(z)$).

Write the series forms from the Mellin-Barnes representation in Excercise 1.6.6 and list the conditions for convergence and existence also.

1.6.6. $\frac{\Gamma(1+2\nu)}{\Gamma(\frac{1}{2}+\nu-\mu)}e^{-z/2}z^{\nu+\frac{1}{2}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\Gamma(\frac{1}{2}+\nu-\mu-s)}{\Gamma(1+2\nu-s)}(-z)^{-s}\mathrm{d}s$ (Whittaker function $M_{\mu,\nu}(z)$)

Represent the following in Excercises 1.6.7 to 1.6.10 as G-functions and write down the conditions.

1.6.7. $z^{\beta}(1+az^{\alpha})^{-1}$

1.6.8. $z^{\beta}(1+az^{\alpha})^{-\gamma}$

1.6.9. $(a)\ \sin z$; $(b)\ \cos z$; $(c)\ \sinh z$; $(d)\ \cosh z$

1.6.10. $(a)\ \ln(1\pm z)$; $(b)\ \ln\left(\frac{1+z}{1-z}\right)$.

1.7. The H-function

This function is a generalization of the G-function. This was available in the literature as a Mellin -Barnes integral but Charles Fox made a detailed study of it in 1960's and hence the function is called Fox's H-function. The Mellin-Barnes representation is the following:

Notation 1.7.1. **H-function**

$$H_{p,q}^{m,n}\left[z\Big|_{(b_1,\beta_1),\cdots,(b_q,\beta_q)}^{(a_1,\alpha_1),\cdots,(a_p,\alpha_p)}\right] = H_{p,q}^{m,n}\left[z\Big|_{[(b_q,\beta_q)]_q}^{[(a_p,\alpha_p)]}\right] = H_{p,q}^{m,n}(z) = H(z).$$

Definition 1.7.1.

$$H_{p,q}^{m,n}\left[z\Big|_{(b_1,\beta_1),\cdots,(b_q,\beta_q)}^{(a_1,\alpha_1),\cdots,(a_p,\alpha_p)}\right] = \frac{1}{2\pi i}\int_L \frac{\left\{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\right\}\left\{\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)\right\}}{\left\{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\right\}\left\{\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)\right\}}z^{-s}\mathrm{d}s \tag{1.7.1}$$

where $\alpha_1,\cdots,\alpha_p,\beta_1,\cdots,\beta_q$ are real positive numbers (integers, rationals or irrationals), a_j's and b_j's are, in general, complex quantities, $i=\sqrt{-1}$ and the contour L seperates the poles of $\Gamma(b_j+\beta_j s)$, $j=1,\cdots,m$ from those of $\Gamma(1-a_j-\alpha_j s)$, $j=1,\cdots,n$. Three paths L, similar to the ones for a G-function, can be given for the H-function also. Details of the existence conditions, various properties and applications may be seen from Mathai and

Saxena (1978) and Mathai (1993). A simplified set of existence conditions is the following: Let,

$$\mu = \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j \text{ and } \beta = \left\{\prod_{j=1}^{q}\alpha_j^{\alpha_j}\right\}\left\{\prod_{j=1}^{q}\beta_j^{-\beta_j}\right\}. \tag{1.7.2}$$

The H-function exists for the following cases:

$$\begin{aligned} &\text{(i)} \quad q \geq 1, \mu > 0, \text{ for all } z, z \neq 0 \\ &\text{(ii)} \quad q \geq 1, \mu = 0, \text{ for } |z| < \beta^{-1} \\ &\text{(iii)} \quad p \geq 1, \mu < 0, \text{ for all } z, z \neq 0 \\ &\text{(iv)} \quad p \geq 1, \mu = 0, \text{ for } |z|, z > \beta^{-1}. \end{aligned} \tag{1.7.3}$$

Two special cases, which follow from the definition itself, may be noted. When $\alpha_1 = 1 = \cdots = \alpha_p = \beta_1 = 1 = \cdots = \beta_q$ then the H-function reduces to a G-function. When all the α_j's and β_j's are rational numbers, that is ratios of two positive integers since by definition the α_j's and β_j's are positive real numbers, we may make a transformation $\frac{s}{u} = s_1$ where u is the common denominator for all the $\alpha_j, j = 1, \cdots, p$ and $\beta_j, j = 1, \cdots, q$. Under this transformation each coefficient of s_1 in each gamma in (1.7.1) becomes a positive integer. Then we may expand all the gammas by using the multiplication formula for gamma functions. Then the coefficients of s in every gamma becomes ± 1 and then the H-function becomes a G-function. An illustration of this aspect was seen in Example 1.6.3.

Example 1.7.1. Evaluate the following reaction rate integral in physics and write it as an H-function.

$$I(\alpha, a, b, \rho) = \int_0^\infty x^{\alpha-1} e^{-ax-bx^{-\rho}} dx.$$

Solution: From (1.6.16) in Example 1.6.3 we have

$$\int_0^\infty x^{\alpha-1} e^{-ax-bx^{-\rho}} dx = \frac{a^{-\alpha}}{\rho}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(\alpha+s)\Gamma(\frac{s}{\rho})(ab^{1/\rho})^{-s} ds. \tag{1.7.4}$$

Writing the right side with the help of (1.7.1) we have the following:

$$\int_0^\infty x^{\alpha-1} e^{-ax-bx^{-\rho}} dx = \frac{1}{\rho a^\alpha} H_{0,2}^{2,0}\left[ab^{\frac{1}{\rho}}\Big|_{(\alpha,1),(0,\frac{1}{\rho})}\right]. \tag{1.7.5}$$

Example 1.7.2. Let $x_1, \cdots, x_k$ be independently distributed real scalar gamma random variables with the parameters $(\alpha_j, 1)$, $j = 1, \cdots, k$. Let $\gamma_1, \cdots, \gamma_k$ be real constants. Let

$$u = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_k^{\gamma_k}.$$

Evaluate the density of u.

Solution: Let us take the $(s-1)^{th}$ moment of u or the Mellin transform of the density of u.

$$E(u^{s-1}) = E[x_1^{\gamma_1} \cdots x_k^{\gamma_k}]^{s-1} = E(x_1^{\gamma_1(s-1)}) \cdots E[x_k^{\gamma_k(s-1)}]$$

due to independence. But for a real gamma random variable, with parameters $(\alpha_j, 1)$, the $[\gamma_j(s-1)]^{th}$ moment is the following:

$$E[x_j^{\gamma_j(s-1)}] = \frac{\Gamma(\alpha_j + \gamma_j(s-1))}{\Gamma(\alpha_j)} = \frac{\Gamma(\alpha_j - \gamma_j + \gamma_j s)}{\Gamma(\alpha_j)} \text{ for } \Re(\alpha_j + \gamma_j(s-1)) > 0. \tag{1.7.6}$$

Then

$$E(u^{s-1}) = \prod_{j=1}^{k} \frac{\Gamma(\alpha_j - \gamma_j + \gamma_j s)}{\Gamma(\alpha_j)}.$$

The density of u, denoted by $g(u)$, is available from the inverse Mellin transform. That is,

$$\begin{aligned} g(u) &= \frac{1}{\left\{\prod_{j=1}^{k} \Gamma(\alpha_j)\right\}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left\{\prod_{j=1}^{k} \Gamma(\alpha_j - \gamma_j + \gamma_j s)\right\} u^{-s} \mathrm{d}s \\ &= \begin{cases} \frac{1}{\left\{\prod_{j=1}^{k} \Gamma(\alpha_j)\right\}} H_{0,k}^{k,0}\left[u\big|_{(\alpha_j - \gamma_j, \gamma_j), j=1,\cdots,k}\right], u > 0, \\ 0, \quad \text{elsewhere.} \end{cases} \end{aligned}$$

This is the density function for the product of arbitrary powers of independently distributed real scalar gamma random variables.

By using similar procedures one can obtain and write in terms of H-functions, products of arbitrary powers of real scalar type-1 beta and type-2 beta random variables or arbitrary powers of products and ratios of real scalar gamma, type-1, type-2 beta or other such positive variables. Some details may by seen from Mathai (1993) and Mathai and Saxena (1978).

Exercises 1.7.

1.7.1. Prove that

$$H_{p,q}^{m,n}[z|_{(b_1,\beta_1),\cdots,(b_q,\beta_q)}^{(a_1,\alpha_1),\cdots,(a_p,\alpha_p)}] = H_{q,p}^{n,m}[\frac{1}{z}|_{(1-a_1,\alpha_1),\cdots,(1-a_p,\alpha_p)}^{(1-b_1,\beta_1),\cdots,(1-b_q,\beta_q)}].$$

1.7.2. Prove that

$$z^{\sigma}H_{p,q}^{m,n}[z|_{(b_1,\beta_1),\cdots,(b_q,\beta_q)}^{(a_1,\alpha_1),\cdots,(a_p,\alpha_p)}] = H_{p,q}^{m,n}[z|_{(b_1+\sigma\beta_1,\beta_1),\cdots,(b_q+\sigma\beta_q,\beta_q)}^{(a_1+\sigma\alpha_1,\alpha_1),\cdots,(a_p+\sigma\alpha_p,\alpha_p)}].$$

1.7.3. Evaluate the Mellin-Barnes integral

$$E_{\alpha}(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)}{\Gamma(1-\alpha s)}(-z)^{-s}\mathrm{d}s \tag{1.7.7}$$

and show that $E_{\alpha}(z)$ is the Mittag-Leffler series

$$E_{\alpha}(z) = \sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\alpha r+1)}. \tag{1.7.8}$$

1.7.4. Evaluate the Laplace transform of $E_{\alpha}(z^{\alpha})$ of Excercise 1.7.3, in (1.7.8), with parameter p.

1.7.5. A generalization of Mittag-Leffler function $E_{\alpha,\beta}(z) = \sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\alpha r+\beta)}$. Evaluate the Laplace transform of $t^{\beta-1}E_{\alpha,\beta}(z^{\alpha})$.

1.7.6. If $\alpha = m, m = 1, 2, \cdots$ in (1.7.8) show that

$$E_{\alpha}(z) = (2\pi)^{\frac{m-1}{2}}m^{-\frac{1}{2}}{}_0F_{m-1}(;\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m};\frac{z}{m^m})\frac{1}{\Gamma(\frac{1}{m}),\Gamma(\frac{2}{m}),\cdots\Gamma(\frac{m-1}{m})}.$$

1.7.7. Write $E_{\alpha}(z)$ as an H-function.

1.7.8. If $\alpha = m, m = 1, 2, \cdots$ write down $E_{\alpha}(z)$ as a G-function.

1.7.9. Let x_1 and x_2 be independently distributed real gamma random variables with the parameters $(\alpha, 1), (\alpha+\frac{1}{2}, 1)$ respectively. Let $u = x_1x_2$. Evaluate the density of u and show that the density of u, denoted by $g(u)$, is given by the following:

$$g(u) = \frac{2^{2\alpha-1}}{\Gamma(2\alpha)}u^{\alpha-1}\mathrm{e}^{-2u^{\frac{1}{2}}}, u \geq 0 \quad \text{and zero elsewhere.}$$

1.7.10. Let x_1, x_2, x_3 be independently distributed real gamma random variables with the parameters $(\alpha, 1), (\alpha + \frac{1}{3}, 1), (\alpha + \frac{2}{3}, 1)$ respectively. Let $u = x_1x_2x_3$. Evaluate the density of u and show that it can be written as an H-function of the following type, where $g(u)$ denotes the density of u.

$$g(u) = \frac{27}{\Gamma(3\alpha)} H^{1,0}_{0,1}[27u\big|_{(3\alpha-3,3)}], u \geq 0 \text{ and zero elsewhere.}$$

1.8. Dirichlet Integrals and Dirichlet Densities

A multivariate integral, which is a generalization of a beta integral, is the Dirichlet integral. We looked at type-1 and type-2 beta integrals. Here we consider type-1 and type-2 Dirichlet integrals and their generalizations. Analogously we will also define the corresponding statistical densities.

Notation 1.8.1. Dirichlet Function: $D(\alpha_1, \cdots, \alpha_k; \alpha_{k+1})$ (real scalar case)

Definition 1.8.1.

$$D(\alpha_1, \cdots, \alpha_k; \alpha_{k+1}) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_{k+1})}{\Gamma(\alpha_1 + \cdots + \alpha_{k+1})} \text{ for } \Re(\alpha_j) > 0, j = 1, \cdots, k+1. \tag{1.8.1}$$

Note that for $k = 1$ we have the beta function in the real scalar case. Consider the following integral:

$$D_1 = \int_\Omega \cdots \int x_1^{\alpha_1 - 1}, \cdots, x_k^{\alpha_k - 1}(1 - x_1 - \cdots - x_k)^{\alpha_{k+1}-1} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k \tag{1.8.2}$$

where $\Omega = \{(x_1, \cdots, x_k)|0 \leq x_i \leq 1, i = 1, \cdots, k, 0 \leq x_1 + \cdots + x_k \leq 1\}$. Since $1 - x_1 - \cdots - x_k \geq 0$ we have $0 \leq x_1 \leq 1 - x_2 - \cdots - x_k$. Integration over x_1 yields the following:

$$\begin{aligned}\int_{x_1=0}^{1-x_2-\cdots-x_k} & x_1^{\alpha_1-1}(1 - x_1 - x_2 - \cdots - x_k)^{\alpha_{k+1}-1}\mathrm{d}x_1 \\ &= (1 - x_2 - \cdots - x_k)^{\alpha_{k+1}-1} \\ &\times \int_{x_1=0}^{1-x_2-\cdots-x_k} x_1^{\alpha_1-1}\Big[1 - \frac{x_1}{1 - x_2 - \cdots - x_k}\Big]^{\alpha_{k+1}-1}\mathrm{d}x_1.\end{aligned}$$

Put

$$y_1 = \frac{x_1}{1 - x_2 - \cdots - x_k} \Rightarrow \mathrm{d}x_1 = (1 - x_2 - \cdots - x_k)\mathrm{d}y_1.$$

Then the integral over x_1 yields,

$$(1 - x_2 - \cdots - x_k)^{\alpha_1+\alpha_{k+1}-1} \int_0^1 y_1^{\alpha_1-1}(1 - y_1)^{\alpha_{k+1}-1}\mathrm{d}y_1$$
$$= (1 - x_2 - \cdots - x_k)^{\alpha_1+\alpha_{k+1}-1} \frac{\Gamma(\alpha_1)\Gamma(\alpha_{k+1})}{\Gamma(\alpha_1 + \alpha_{k+1})}$$

for $\Re(\alpha_1) > 0$, $\Re(\alpha_{k+1}) > 0$. Integral over x_2 yields,

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_{k+1})}{\Gamma(\alpha_1 + \alpha_{k+1})} \frac{\Gamma(\alpha_2)\Gamma(\alpha_1 + \alpha_{k+1})}{\Gamma(\alpha_1 + \alpha_2 + \alpha_{k+1})} = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_{k+1})}{\Gamma(\alpha_1 + \alpha_2 + \alpha_{k+1})}.$$

Proceeding like this, we have the final result:

$$D_1 = D(\alpha_1, \cdots, \alpha_k; \alpha_{k+1}) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_{k+1})}{\Gamma(\alpha_1 + \cdots + \alpha_{k+1})}, \ \Re(\alpha_j) > 0, j = 1, \cdots, k + 1. \tag{1.8.3}$$

Here, (1.8.2) is the type-1 Dirichlet integral. Hence by normalizing the integrand in (1.8.2) we have the type-1 Dirichlet density.

Definition 1.8.2. Type-1 Dirichlet density $f_1(x_1, \cdots, x_k)$**.**

$$f_1(x_1, \cdots, x_k) = \frac{1}{D(\alpha_1, \cdots, \alpha_k; \alpha_{k+1})} x_1^{\alpha_1-1} \cdots x_k^{\alpha_k-1}(1 - x_1 - \cdots - x_k)^{\alpha_{k+1}-1},$$
$$0 \le x_j \le 1, j = 1, \cdots, k, \ 0 \le x_1 + \cdots + x_k \le 1, \ \Re(\alpha_j) > 0, \tag{1.8.4}$$
$$j = 1, \cdots, k + 1, \text{ and } f_1(x_1, \cdots, x_k) = 0 \text{ elsewhere.}$$

Consider the type-2 Dirichlet integral

$$D_2 = \int_0^\infty \cdots \int_0^\infty x_1^{\alpha_1-1} \cdots x_k^{\alpha_k-1}(1 + x_1 + \cdots + x_k)^{-(\alpha_1+\cdots+\alpha_{k+1})}\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k. \tag{1.8.5}$$

This can be integrated by writing

$$(1 + x_1 + \cdots + x_k) = (1 + x_2 + \cdots + x_k)\left[1 + \frac{x_1}{1 + x_2 + \cdots + x_k}\right]$$

and then integrating out with the help of type-2 beta integrals. The final result will agree with the Dirichlet function

$$D_2 = D(\alpha_1, \cdots \alpha_k; \alpha_{k+1}). \tag{1.8.6}$$

Thus, we can define a type-2 Dirichlet density.

Definition 1.8.3. Type-2 Dirichlet density.

$$f_2(x_1, \cdots, x_k) = \frac{1}{D(\alpha_1, \cdots, \alpha_k; \alpha_{k+1})} x_1^{\alpha_1 - 1} \cdots x_k^{\alpha_k - 1} (1 + x_1 + \cdots + x_k)^{-(\alpha_1 + \cdots + \alpha_{k+1})},$$
$$0 \le x_j < \infty, \ \ j = 1, \cdots, k, \ \ \Re(\alpha_j) > 0, \ \ j = 1, \cdots, k+1, \tag{1.8.7}$$

and $f_2(x_1, \cdots, x_k) = 0$ elsewhere.

It is easy to observe that if $(x_1, \cdots, x_k)$ has a k - variate type-1 Dirichlet density then any subset of r of the variables have a r - variate type-1 Dirichlet density for $r = 1, \cdots, k$. Similarly if $(x_1, \cdots, x_k)$ have a type-2 Dirichlet density then any subset of them will have a type-2 Dirichlet density.

Example 1.8.1. Evaluate the marginal densities from the following bivariate density:

$$f(x_1, x_2) = \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} (1 - x_1 - x_2)^{\alpha_3 - 1}, 0 \le x_j \le 1, j = 1, 2, 3,$$
$$0 \le x_1 + x_2 + x_3 \le 1, \ \Re(\alpha_j) > 0, j = 1, 2, 3, \text{ and } f(x_1, x_2) = 0 \text{ elsewhere}.$$

Solution: Let the marginal densities be denoted by $f_1(x_1)$ and $f_2(x_2)$ respectively.

$$\begin{aligned} f_1(x_1) &= \int_{x_2} f(x_1 x_2) \mathrm{d}x_2 = \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x_1^{\alpha_1 - 1} \\ &\times \int_{x_2 = 0}^{1 - x_1} x_2^{\alpha_2 - 1} (1 - x_1 - x_2)^{\alpha_3 - 1} \mathrm{d}x_2 \\ &= \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x_1^{\alpha_1 - 1} (1 - x_1)^{\alpha_3 - 1} \int_{x_2 = 0}^{1 - x_1} x_2^{\alpha_2 - 1} \left[1 - \frac{x_2}{1 - x_1}\right]^{\alpha_3 - 1} \mathrm{d}x_2. \end{aligned}$$

Put

$$y_2 = \frac{x_2}{1-x_1} \Rightarrow \mathrm{d}x_2 = (1-x_1)\mathrm{d}y_2.$$

$$f_1(x_1) = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x_1^{\alpha_1-1}(1-x_1)^{\alpha_2+\alpha_3-1}\int_0^1 y_2^{\alpha_2-1}(1-y_2)^{\alpha_3-1}\mathrm{d}y_2.$$

Evaluating the y_2-integral with the help of a type-1 beta integral we obtain $\frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_2+\Gamma\alpha_3)}$. Hence,

$$f_1(x_1) = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2+\alpha_3)} x_1^{\alpha_1-1}(1-x_1)^{\alpha_2+\alpha_3-1}, 0 \le x_1 \le 1,$$

and zero elsewhere. From symmetry,

$$f_2(x_2) = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_2)\Gamma(\alpha_1+\alpha_3)} x_2^{\alpha_2-1}(1-x_2)^{\alpha_1+\alpha_3-1}\ 0 \le x_2 \le 1,$$

and zero elsewhere. Thus, the marginal densities of x_1 and x_2 are type-1 beta densities.

Example 1.8.2. Evaluate the normalizing constant c if the following is a density function:

$$f(x_1,x_2) = cx_1^{\alpha_1-1}(1-x_1)^{\beta_1}x_2^{\alpha_2-1}(1-x_1-x_2)^{\alpha_3-1}, 0 \le x_j \le 1, \tag{1.8.8}$$

$0 \le x_1+x_2 \le 1$, $j = 1,2$, $\Re(\alpha_j) > 0$, $j = 1,2,3$ and $f(x_1,x_2) = 0$ elsewhere.

Solution: Let us integrate out x_2 first.

$$\begin{aligned}\int_{x_2=0}^{1-x_1} & x_2^{\alpha_2-1}(1-x_1-x_2)^{\alpha_3-1}\mathrm{d}x_2 \\ &= (1-x_1)^{\alpha_2+\alpha_3-1}\int_0^1 y_2^{\alpha_2-1}(1-y_2)^{\alpha_3-1}\mathrm{d}y_2, y_2 = \frac{x_2}{1-x_1} \\ &= (1-x_1)^{\alpha_2+\alpha_3-1}\frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_2+\alpha_3)},\ \ \Re(\alpha_2) > 0,\ \ \Re(\alpha_3) > 0.\end{aligned}$$

Now, integrating out x_1 we have,

$$\int_0^1 x_1^{\alpha_1-1}(1-x_1)^{\beta_1+\alpha_2+\alpha_3-1}\mathrm{d}x_1 = \frac{\Gamma(\alpha_1)\Gamma(\beta_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\beta_1)}$$

$\Re(\alpha_1) > 0$, $\Re(\beta_1+\alpha_2+\alpha_3) > 0$. Hence

$$c = \frac{\Gamma(\alpha_2+\alpha_3)\Gamma(\alpha_1+\alpha_2+\alpha_3+\beta_1)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_2+\alpha_3+\beta_1)}, \quad \Re(\alpha_j)>0,\ j=1,2,3, \Re(\alpha_2+\alpha_3+\beta_1)>0.$$

A generalization to k variable case is one of the generalizations of type-1 Dirichlet density and the corresponding type-1 Dirichlet function.

Example 1.8.3. Evaluate the normalizing constant if the following is a density function:

$$\begin{aligned} f(x_1,x_2,x_3) &= cx_1^{\alpha_1-1}(x_1+x_2)^{\beta_2}\ x_2^{\alpha_2-1}(x_1+x_2+x_3)^{\beta_3}\ x_3^{\alpha_3-1}(1-x_1-x_2-x_3)^{\alpha_4-1},\\ &0\le x_1+\cdots+x_j\le 1,\quad j=1,2,3,4,\quad \Re(\alpha_j)>0,\quad j=1,2,3,4, \qquad (1.8.9)\\ &\Re(\alpha_1+\cdots+\alpha_j+\beta_2+\cdots+\beta_j)>0, j=1,2,3,4 \end{aligned}$$

and $f(x_1,x_2,x_3)=0$ elsewhere.

Solution: Let $u_1=x_1,\ u_2=x_1+x_2\ u_3=x_1+x_2+x_3$ and let the joint density of u_1,u_2,u_3 be denoted by $g(u_1,u_2,u_3)$. Then

$$\begin{aligned} g(u_1,u_2,u_3) &= c\ u_1^{\alpha_1-1}u_2^{\beta_2}(u_2-u_1)^{\alpha_2-1}u_3^{\beta_3}(u_3-u_2)^{\alpha_3-1}(1-u_3)^{\alpha_4-1},\\ &0\le u_1\le u_2\le u_3\le 1. \end{aligned}$$

Note that $0\le u_1\le u_2$. Integration over u_1 yields the following:

$$\begin{aligned} \int_{u_1=0}^{u_2} u_1^{\alpha_1-1}(u_2-u_1)^{\alpha_2-1}\mathrm{d}u_1 &= u_2^{\alpha_2-1}\int_{u_1=0}^{u_2} u_1^{\alpha_1-1}(1-\frac{u_1}{u_2})^{\alpha_2-1}\mathrm{d}u_1\\ &= u_2^{\alpha_1+\alpha_2-1}\int_0^1 y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1}\mathrm{d}y_1,\ y_1=\frac{u_1}{u_2}\\ &= u_2^{\alpha_1+\alpha_2-1}\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)},\ \Re(\alpha_1)>0, \Re(\alpha_2)>0. \end{aligned}$$

Integration over u_2 yields the following:

$$\begin{aligned} \int_{u_2=0}^{u_3} u_2^{\alpha_1+\alpha_2+\beta_2-1}(u_3-u_2)^{\alpha_3-1}\mathrm{d}u_2 &= u_3^{\alpha_1+\alpha_2+\alpha_3+\beta_2-1}\frac{\Gamma(\alpha_3)\Gamma(\alpha_1+\alpha_2+\beta_2)}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\beta_2)}\\ &\text{for } \Re(\alpha_3)>0, \Re(\alpha_1+\alpha_2+\beta_2)>0. \end{aligned}$$

Finally, integral over u_3 yields the following:

$$\int_{u_3=0}^{1} u_3^{\alpha_1+\alpha_2+\alpha_3+\beta_2+\beta_3-1}(1-u_3)^{\alpha_4-1}\mathrm{d}u_3 = \frac{\Gamma(\alpha_4)\Gamma(\alpha_1+\alpha_2+\alpha_3+\beta_2+\beta_3)}{\Gamma(\alpha_1+\cdots+\alpha_4+\beta_2+\beta_3)},$$
$$\Re(\alpha_4)>0, \Re(\alpha_1+\alpha_2+\alpha_3+\beta_2+\beta_3)>0.$$

Hence

$$c^{-1} = \Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_4)\frac{\Gamma(\alpha_1+\alpha_2+\beta_2)}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\beta_2)}$$
$$\times\frac{\Gamma(\alpha_1+\alpha_2+\alpha_3+\beta_2+\beta_3)}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\beta_2+\beta_3)}$$

for $\Re(\alpha_j)>0,\ j=1,2,3,4,\ \Re(\alpha_1+\cdots+\alpha_j+\beta_2+\cdots+\beta_j)>0,\ j=2,3.$

Note that one can generalize the function in (1.8.9) to a k- variables situation. This will produce another generalization of the type-1 Dirichlet function as well as the type-1 Dirichlet density. Corresponding situations in the type-2 case will provide generalizaions of the type-2 Dirichlet integral and density.

Exercises 1.8.

1.8.1. Let $f(x_1,x_2,x_3) = cx_1^{\alpha_1-1}(1+x_1)^{-(\alpha_1+\beta_1)}x_2^{\alpha_2-1}(1+x_1+x_2)^{-(\alpha_2+\beta_2)}x_3^{\alpha_3-1}(1+x_1+x_2+x_3)^{-(\alpha_3+\beta_3)},\ 0\le x_j<\infty,\ j=1,2,3$ and $f(x_1,x_2,x_3)=0$ elsewhere. If $f(x_1,x_2,x_3)$ is a density function then evaluate c and write down the conditions on the parameters.

1.8.2. Generalize the density in Excercise 1.8.1 to k-variables case, evaluate the corresponding c and write down the conditions.

1.8.3. Write down the k-variables situation in Example 1.8.3 and evaluate the normalizing constant, and give the conditions on the parameters.

1.8.4. Write down the general density corresponding to Example 1.8.2 and evaluate the normalizing constant, and give the conditions on the parameters.

1.8.5. By using the gamma structure in the normalizing constant in Exercise 1.8.4 show that the joint density in Exercise 1.8.4 can also be obtained as the joint density of k mutually independently distributed real scalar type-1 beta random variables, and identify the parameters in these independent type-1 beta random variables.

1.9. Lauricella Functions and Appell's Functions

Another set of multivariable functions in frequent use in applied areas is the set of Lauricalla functions, and special cases of those are the Appell's functions. Lauricella functions f_A, f_B, f_C, and f_D are the following:

Definition 1.9.1. Lauricella function f_A

$$f_A(a, b_1, \cdots, b_n; c_1, \cdots, c_n; x_1, \cdots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1} \cdots (b_n)_{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!} \tag{1.9.1}$$

for $|x_1| + \cdots + |x_n| < 1$.

Definition 1.9.2. Lauricella function f_B

$$f_B(a_1, \cdots, a_n, b_1, \cdots, b_n; c; x_1, \cdots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a_1)_{m_1} \cdots (a_n)_{m_n}(b_1)_{m_1} \cdots (b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!} \tag{1.9.2}$$

for $|x_1| < 1, |x_2| < 1, \cdots, |x_n| < 1$.

Definition 1.9.3. Lauricella function f_C

$$f_C(a, b; c_1, \cdots, c_n; x_1, \cdots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b)_{m_1+\cdots+m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!} \tag{1.9.3}$$

for $|\sqrt{x_1}| + \cdots + |\sqrt{x_n}| < 1$.

Definition 1.9.4. Lauricella function f_D

$$f_D(a, b_1, \cdots, b_n; c; x_1, \cdots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1} \cdots\cdots (b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!} \tag{1.9.4}$$

for $|x_1| < 1, |x_2| < 1, \cdots, |x_n| < 1$.

When $n = 2$ we have Appell's functions F_1, F_2, F_3, F_4. Also when $n = 1$ all these functions reduce to a Gauss' hypergeometric function ${}_2F_1$. We will list some of the basic properties of Lauricella functions.

1.9.1. Some properties of f_A

$$\int_0^1 \cdots \int_0^1 u_1^{b_1-1} \cdots u_n^{b_n-1}(1-u_1)^{c_1-b_1-1} \cdots (1-u_n)^{c_n-b_n-1} \times (1-u_1x_1-\cdots-u_nx_n)^{-a}\, du_1 \wedge \cdots \wedge du_n = \left\{\prod_{j=1}^{n} \frac{\Gamma(b_j)\Gamma(c_j-b_j)}{\Gamma(c_j)}\right\} f_A(a, b_1, \cdots, b_n; c_1, \cdots, c_n; x_1, \cdots, x_n), \tag{1.9.5}$$

for $\Re(b_j) > 0, \ \Re(c_j - b_j) > 0, j = 1, \cdots, n$.

The result can be easily established by expanding the factor $(1 - u_1x_1 - \cdots - u_nx_n)^{-a}$ by using a multinomial expansion and then integrating out $u_j, j = 1, \cdots, n$ with the help of type-1 beta integrals.

$$\int_0^\infty e^{-t}t^{a-1}{}_1F_1(b_1; c_1; x_1t){}_1F_1(b_2; c_2; x_2t) \cdots {}_1F_1(b_n; c_n; x_nt)dt = \Gamma(a) f_A(a, b_1, \cdots, b_n; c_1, \cdots, c_n; x_1, \cdots, x_n), \text{ for } \Re(a) > 0. \tag{1.9.6}$$

This can be established by taking the series forms for ${}_1F_1$'s and then integrating out t.

$$\frac{1}{(2\pi i)^n} \int \cdots \int \frac{\Gamma(a+t_1+\cdots+t_n)\Gamma(b_1+t_1)\cdots\Gamma(b_n+t_n)}{\Gamma(c_1+t_1)\cdots\Gamma(c_n+t_n)} \times \Gamma(-t_1)\cdots\Gamma(-t_n)(-x_1)^{t_1}\cdots(-x_n)^{t_n}\, dt_1 \wedge \cdots \wedge dt_n = \Gamma(a)\frac{\Gamma(b_1)\cdots\Gamma(b_n)}{\Gamma(c_1)\cdots\Gamma(c_n)} f_A(a, b_1, \cdots, b_n; c_1, \cdots, c_n; x_1, \cdots, x_n), i = \sqrt{-1}. \tag{1.9.7}$$

This can be established by evaluating the integrand as the sum of the residues at the poles of $\Gamma(-t_1), \cdots, \Gamma(-t_n)$, one by one.

1.9.2. Some properties of f_B

$$\int\cdots\int t_1^{a_1-1}\cdots t_n^{a_n-1}(1-t_1-\cdots-t_n)^{c-a_1-\cdots-a_n-1} \times(1-t_1x_1)^{-b_1}\cdots(1-t_nx_n)^{-b_n}\,\mathrm{d}t_1\wedge\cdots\wedge\mathrm{d}t_n = \frac{\Gamma(a_1)\cdots\Gamma(a_n)\Gamma(c-a_1-\cdots-a_n)}{\Gamma(c)} f_B(a_1,\cdots,a_n,b_1,\cdots,b_n;c;x_1,\cdots,x_n), \tag{1.9.8}$$

for $\Re(a_j)>0, j=1,\cdots,n, \Re(c-a_1-\cdots-a_n)>0, t_j>0, j=1,\cdots,n,$ and $1-t_1-\cdots-t_n>0$.

This result can be established by opening up $(1-t_jx_j)^{-b_j}$, $j=1,\cdots,n$ by using binomial expansions and then integrating out $t_1,\cdots,t_n$ with the help of a type-1 Dirichlet integral of Section 1.8.

$$\int_0^\infty\cdots\int_0^\infty s_1^{a_1-1}\cdots s_n^{a_n-1}t_1^{b_1-1}\cdots t_n^{b_n-1}\mathrm{e}^{-s_1-\cdots-s_n-t_1-\cdots-t_n} \times {}_0F_1(;c;s_1t_1x_1+\cdots+s_nt_nx_n)\,\mathrm{d}s_1\wedge\cdots\wedge\mathrm{d}s_n\wedge\mathrm{d}t_1\wedge\cdots\wedge\mathrm{d}t_n = \left\{\prod_{j=1}^n\Gamma(a_j)\Gamma(b_j)\right\} f_B(a_1\cdots,a_n,b_1,\cdots,b_n;c;x_1,\cdots,x_n), \tag{1.9.9}$$

for $\Re(a_j)>0, \quad \Re(b_j)>0, j=1,\cdots,n.$

First, open up the ${}_0F_1$ as a power series in $(s_1t_1x_1+\cdots+s_nt_nx_n)^k$. Since k is a positive integer open up by using a multinomial expansion. Then integrate out $s_1,\cdots,s_k$ and $t_1,\cdots,t_k$ by using gamma functions, to see the result.

$$\frac{1}{(2\pi i)^n}\int\cdots\int\frac{\Gamma(a_1+t_1)\cdots\Gamma(a_n+t_n)\Gamma(b_1+t_1)\cdots\Gamma(b_n+t_n)}{\Gamma(c+t_1+\cdots+t_n)} \times\Gamma(-t_1)\cdots\Gamma(-t_n)(-x_1)^{t_1}\cdots(-x_n)^{t_n}\,\mathrm{d}t_1\wedge\cdots\wedge\mathrm{d}t_n,\ i=\sqrt{-1} = \left\{\prod_{j=1}^n\frac{\Gamma(a_j)\Gamma(b_j)}{\Gamma(c)}\right\} f_B(a_1,\cdots,a_n,b_1,\cdots,b_n;c;x_1,\cdots,x_n). \tag{1.9.10}$$

Assume that $a_j-b_j\neq\pm\nu,\nu=0,1,\cdots$. Then evaluate the integrand as the sum of the residues at the poles of $\Gamma(-t_1),\cdots,\Gamma(-t_n)$, one by one, to obtain the result.

1.9.3. Some properties of f_C

$$\int_0^\infty \cdots \int_0^\infty s^{a-1} t^{b-1} \mathrm{e}^{-s-t} \, {}_0F_1\,(;c_1;x_1 st) \cdots {}_0F_1\,(;c_n;x_n st) \mathrm{d}s \wedge \mathrm{d}t \tag{1.9.11}$$
$$= \Gamma(a)\Gamma(b) f_C(a,b;c_1,\cdots,c_n;x_1,\cdots,x_n), \quad \text{for } \Re(a) > 0, \Re(b) > 0.$$

Open up the ${}_0F_1$'s, then integrate out t and s with the help of gamma integrals to see the result.

$$\frac{1}{(2\pi i)^n} \int \cdots \int \frac{\Gamma(a+t_1+\cdots+t_n)\Gamma(b+t_1+\cdots+t_n)}{\Gamma(c_1+t_1)\cdots\Gamma(c_n+t_n)}$$
$$\times\ \Gamma(-t_1)\cdots\Gamma(-t_n)(-x_1)^{t_1}\cdots(-x_n)^{t_n} \mathrm{d}t_1, \wedge \cdots \wedge \mathrm{d}t_n$$
$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c_1)\cdots\Gamma(c_n)} f_C(a,b;c_1,\cdots,c_n;x_1,\cdots,x_n), i = \sqrt{-1}. \tag{1.9.12}$$

Evaluate the integrand as the sum of the residues at the poles of $\Gamma(-t_1),\cdots,\Gamma(-t_n)$, one by one, to obtain the result.

1.9.4. Some properties of f_D

$$\int \cdots \int u_1^{b_1-1} \cdots u_n^{b_n-1} (1-u_1 \cdots u_n)^{c-b_1-\cdots-b_n-1} \tag{1.9.13}$$
$$\times (1-u_1 x_1 - \cdots - u_n x_n)^{-a} \mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n$$
$$= \frac{\Gamma(b_1)\cdots\Gamma(b_n)\Gamma(c-b_1-\cdots-b_n)}{\Gamma(c)} f_D(a,b_1,\cdots,b_n;c;x_1,\cdots,x_n), \text{ for}$$
$$0 < u_j < 1,\ j=1,\cdots,n,\ 0 < u_1+\cdots+u_n < 1,\ 0 < x_1 u_1 + \cdots + x_n u_n < 1,$$
$$\Re(b_j) > 0,\ j=1,\cdots,n,\ \Re(c-b_1-\cdots-b_n) > 0.$$

Open up $(1-u_1x_1-\cdots-u_nx_n)^{-a}$ by using a multinomial expansion and then integrate out $u_1,\cdots,u_n$ by using a type-1 Dirichlet integral of Section 1.8.

$$\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux_1)^{-b_1}\cdots(1-ux_n)^{-b_n}\mathrm{d}u \tag{1.9.14}$$
$$= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} f_D(a,b_1,\cdots,b_n;c;x_1,\cdots,x_n) \text{ for } \Re(a) > 0,\ \Re(c-a) > 0.$$

Expand $(1-ux_j)^{-b_j}$, $j=1,\cdots,n$ by using binomial expansions and then integrate out u by using a type-1 beta integral to see the result.

$$\int_0^\infty \cdots \int_0^\infty t_1^{b_1-1} \cdots t_n^{b_n-1} \mathrm{e}^{-t_1-\cdots-t_n} {}_1F_1(a; c; x_1 t_1 + \cdots + x_n t_n) \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n \tag{1.9.15}$$
$$= \Gamma(b_1) \cdots \Gamma(b_n) f_D(a, b_1, \cdots, b_n; c; x_1, \cdots, x_n), \text{ for } \Re(b_j) > 0, j = 1, \cdots, n.$$

Expand ${}_1F_1$ as a series, then open up the general term with the help of a multinomial expansion for positive integral exponent, then integrate out $t_1, \cdots, t_n$ to see the result.

$$\frac{1}{(2\pi i)^n} \int \cdots \int \frac{\Gamma(a + t_1 + \cdots + t_n)\Gamma(b_1 + t_1) \cdots \Gamma(b_n + t_n)}{\Gamma(c + t_1 + \cdots + t_n)} \tag{1.9.16}$$
$$\times \Gamma(-t_1) \cdots \Gamma(-t_n)(-x_1)^{t_1} \cdots (-x_n)^{t_n} \mathrm{d}t_1, \wedge \cdots \wedge \mathrm{d}t_n$$
$$= \frac{\Gamma(a)\Gamma(b_1) \cdots \Gamma(b_n)}{\Gamma(c)} f_D(a, b_1, \cdots, b_n; c; x_1, \cdots, x_n), i = \sqrt{-1}.$$

Follow through the same method of evaluation of the contour integrals as in f_A, f_B and f_C to see the result.

$$f_D(a, b_1, \cdots, b_n; c; x, \cdots, x) = {}_2F_1(a, b_1 + \cdots + b_n; c; x). \tag{1.9.17}$$

Use the integral representaion in (1.9.14) and put $x_1 = \cdots = x_n = x$ to see the result.

$$f_D(a, b_1, \cdots, b_n; c; 1, 1 \cdots, 1) = \frac{\Gamma(c)\Gamma(c - a - b_1 - \cdots - b_n)}{\Gamma(c - a)\Gamma(c - b_1 - \cdots - b_n)}. \tag{1.9.18}$$

Evaluate (1.9.17) at $x = 1$ to see the result.

There are other functions in the category of multivariable hypergeometric functions known as Humbert's functions, Kampé de Fériet functions and so on. These will not be discussed here. For a brief description of these, along with some of their properties, see for example Mathai (1993, 1997) and Srivasthava and Karlsson (1985).

Example 1.9.1. Show that

$$f_A(a, b_1, \cdots, b_n; c_1; \cdots, c_n; x_1, \cdots, x_n) \tag{1.9.19}$$
$$= \sum_{m_1=0}^{\infty} \cdots, \sum_{m_{n-1}=0}^{\infty} \frac{(a)_{m_1+\cdots+m_{n-1}}(b)_{m_1} \cdots (b_n)_{m_n}}{(a)_{m_1} \cdots (c_n)_{m_n}}$$
$$\times \frac{x_1^{m_1} \cdots x_{n-1}^{m_{n-1}}}{m_1! \cdots m_{n-1}!} {}_2F_1(a + m_1 + \cdots + m_{n-1}, b_n; c_n; x_n), |x_1| + \cdots + |x_n| < 1.$$

Solution: This can be seen by summing up with respect to m_n by observing that $(a)_{m_1+\cdots+m_n} = (a + m_1 + \cdots + m_{n-1})_{m_n}$. Then the sum is the following:

$$\sum_{m_n=0}^{\infty} \frac{(a+m_1+\cdots+m_{n-1})_{m_n}(b_n)_{m_n}}{(c_n)_{m_n}} \frac{x^{m_n}}{m_n!} = {}_2F_1(a+m_1+\cdots+m_{n-1}, b_n; c_n; x_n).$$

Example 1.9.2. Show that

$$\Gamma(a) f_C\Big(\frac{a}{2}, \frac{a+1}{2}; c_1, \cdots, c_n; x_1, \cdots, x_n\Big) \tag{1.9.20}$$
$$= \int_0^{\infty} t^{a-1} e^{-t} {}_0F_1\Big(; c_1; \frac{t^2 x_1}{4}\Big) \cdots {}_0F_1\Big(; c_n; \frac{t^2 x_n}{4}\Big)\, dt.$$

Solution: Expand the ${}_0F_1$'s. Then the right side becomes,

$$\sum_{m_1=0}^{\infty} \cdots, \sum_{m_n=0}^{\infty} \frac{\int_0^{\infty} t^{a+2m_1+\cdots+2m_n-1}}{(c_1)_{m_1}\cdots(c_n)_{m_n}} e^{-t} \frac{x_1^{m_1}}{4^{m_1}} \cdots \frac{x_n^{m_n}}{4^{m_n}} \frac{1}{m_1!\cdots m_n!}\, dt.$$

Integral over t yields

$$\int_0^{\infty} t^{a+2m_1+\cdots+2m_n-1} e^{-t} dt = \Gamma(a+2m_1+\cdots+2m_n).$$

Expanding $\Gamma(a+2m_1+\cdots+2m_n) = \Gamma[2(\frac{a}{2}+m_1+\cdots+m_n)]$ by using the duplication formula, we have,

$$\begin{aligned}
\Gamma\Big[2\Big(\frac{a}{2}+m_1+\cdots+m_n\Big)\Big] &= \pi^{-\frac{1}{2}} 2^{a+2m_1+\cdots+2m_n-1} \Gamma\Big(\frac{a}{2}+m_1+\cdots+m_n\Big) \\
&\quad \times \Gamma\Big(\frac{a+1}{2}, m_1+\cdots+m_n\Big) \\
&= \pi^{-\frac{1}{2}} 2^{a-1} \Gamma\Big(\frac{a}{2}\Big)\Gamma\Big(\frac{a+1}{2}\Big) \\
&\quad \times \Big(\frac{a}{2}\Big)_{m_1+\cdots+m_n} \Big(\frac{a+1}{2}\Big)_{m_1+\cdots+m_n} (4)^{m_1+\cdots+m_n} \\
&= \Gamma(a) \Big(\frac{a}{2}\Big)_{m_1+\cdots+m_n} \Big(\frac{a+1}{2}\Big)_{m_1+\cdots+m_n} (4)^{m_1+\cdots+m_n}
\end{aligned}$$

(duplication formula is again applied on $\Gamma(a) = \Gamma[2(\frac{a}{2})]$). Now, substituting and interpreting as a f_C the result follows.

Example 1.9.3. Show that $f_B(a_1,\cdots,a_n,b_1,\cdots,b_n;c;x_1,\cdots,x_n)$

$$= \frac{\Gamma(c)}{\Gamma(d_1)\cdots\Gamma(d_n)\Gamma(c-d_1-\cdots-d_n)}\int\cdots\int u_1^{d_1-1}\cdots u_n^{d_n-1} \tag{1.9.21}$$
$$\times(1-u_1-\cdots-u_n)^{c-d_1-\cdots-d_n-1}$$
$$\times\ {}_2F_1(a_1,b_1;d_1;u_1x_1)\cdots{}_2F_1(a_n,b_n;d_n;u_nx_n)du_1\wedge\cdots\wedge du_n$$

for $\Re(\mathrm{d}_j)>0,1,\cdots,n,\quad \Re(c-d_1-\cdots-d_n)>0,|x_j|<1,\ j=1,\cdots,n.$

Solution: Expand the product of ${}_2F_1$'s first.

$$ {}_2F_1(a_1,b_1;d_1;u_1x_1)\cdots{}_2F_1(a_n,b_n;d_n;u_nx_n) \tag{1.9.22}$$
$$=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(a_1)_{m_1}\cdots(a_n)_{m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(d_1)_{m_1}\cdots(d_n)_{m_n}}\frac{(u_1x_1)^{m_1}\cdots(u_nx_n)^{m_n}}{m_1!\cdots m_n!}.$$

Now, evaluate the integral over $u_1,\cdots,u_n$ by using a type-1 Dirichlet integral.

$$\int\cdots\int u_1^{d_1+m_1-1}\cdots u_n^{d_n+m_n-1}(1-u_1-\cdots-u_n)^{c-d_1-\cdots-d_n-1}du_1\wedge\cdots\wedge du_n$$
$$=\frac{\Gamma(d_1+m_1)\cdots\Gamma(d_n+m_n)\Gamma(c-d_1-\cdots-d_n)}{\Gamma(c+m_1+\cdots+m_n)}$$
$$=\frac{\Gamma(c-d_1-\cdots-d_n)\Gamma(d_1)\cdots\Gamma(d_n)}{\Gamma(c)}\frac{(d_1)_{m_1}\cdots(d_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \tag{1.9.23}$$

for $\Re(d_j)>0,\ j=1,\cdots,n,\ \Re(c-d_1-\cdots-d_n)>0$. Now, substituting (1.9.23) and (1.9.22) on the right side of (1.9.21) the result follows.

Exercises 1.9.

1.9.1. Establish the result in (1.9.5)

1.9.2. Establish the result in (1.9.6)

1.9.3. Establish the result in (1.9.7)

1.9.4. Establish the result in (1.9.8)

1.9.5. Establish the result in (1.9.9)

1.9.6. Establish the result in (1.9.10)

1.9.7. Establish the result in (1.9.11)

1.9.8. Establish the result in (1.9.12)

1.9.9. Establish the result in (1.9.13)

1.9.10. Establish the result in (1.9.14)

References

Mathai, A.M. (1993). *A Handbook of Generalized Special Functions for Statistical and Physical Sciences*. Oxford University Press, Oxford.

Mathai, A.M. (1997). *Jacobians of Matrix Transformations and Functions of Matrix Argument*. World Scientific Publishing, New York.

Mathai, A.M. and Saxena, R.K. (1973). *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences*. Springer-Verlag, Lecture Notes 348, Heidelberg.

Mathai, A.M. and Saxena, R.K. (1978). *The H-function with Applications in Statistics and Other Discciplines*. Wiley Halsted, New York.

Srivastava, H.M. and Karlsson, P.W. (1985). *Multiple Gaussian Hypergeometric Series*. Ellis Horwood, Chichester, U.K.