CHAPTER 3

MITTAG-LEFFLER FUNCTIONS AND FRACTIONAL CALCULUS

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3.0. Introduction

This section deals with Mittag-Leffler function and its generalizations. Its importance is realised during the last one and a half decades due to its direct involvement in the problems of physics, biology, engineering and applied sciences. Mittag-Leffler function naturally occurs as the solution of fractional order differential equations or fractional order integral equations. Various properties of Mittag-Leffler functions are described. Among the various results presented by various researchers, the important ones deal with Laplace transform and asymptotic expansions of these functions, which are directly applicable in the solution of differential equations and behavior of the solution for small and large values of the argument. Hille and Tamarkin [14, p.86] in 1920 have presented a solution of Abel-Volterra type integral equation

$$\phi(x) - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{\phi(t)\mathrm{d}t}{(x-t)^{1-\alpha}} = f(x), 0 < x < 1$$

in terms of Mittag-Leffler function. Dzherbashyan [2] has shown that both the functions defined by (3.1.1) and (3.1.2) are entire functions of order $p = \frac{1}{\alpha}$ and type $\sigma = 1$. A detailed account of the basic properties of these functions is given in the third volume of Bateman Manuscript Project written by A. Erdélyi et al [3] and published by McGraw-Hill in the year 1955 under the heading "Miscellaneous Functions".

3.1. Mittag-Leffler Functions

Notation 3.1.1. $E_\alpha(x)$**: Mittag-Leffler function**

Notation 3.1.2. $E_{\alpha,\beta}(x)$**: Generalized Mittag-Leffler function**

Note 3.1.1. : According to Erdélyi, A. et al, $E_\alpha(x)$ and $E_{\alpha,\beta}(x)$ are called Mittag-Leffler functions.

Definition 3.1.1.

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad (\alpha \in C, \ \Re(\alpha) > 0). \tag{3.1.1}$$

Definition 3.1.2.

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha, \beta \in C, \ \Re(\alpha) > 0, \ \Re(\beta) > 0). \tag{3.1.2}$$

The function $E_\alpha(z)$ was defined and studied by Mittag-Leffler in the year 1903. It is a direct generalization of the exponential function. The function defined by (3.1.2) gives a generalization of (3.1.1). This generalization was studied by Wiman in 1905, Agarwal in 1953 and Humbert and Agarwal in 1953, and others.

Example 3.1.1. Prove that $E_1(z) = e^z = E_{1,1}(z)$. It readily follows from (3.1.1) and (3.1.2).

Example 3.1.2. Prove that $E_{1,2}(z) = \frac{e^z - 1}{z}$

Solution: We have

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{1}{z}(e^z - 1).$$

Definition 3.1.3. Hyperbolic function of order n**.**

$$h_r(z, n) := \sum_{k=0}^{\infty} \frac{z^{nk+r-1}}{(nk + r - 1)!} = z^{r-1} E_{n,r}(z^n), \ (r \in N). \tag{3.1.3}$$

Definition 3.1.4. **Trigonometric function of order n.**

$$k_r(z,n) : \quad = \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{kn+r-1}}{(kn+r-1)!} = z^{r-1}E_{n,r}(-z^n). \tag{3.1.4}$$

$$E_{\frac{1}{2},1}(z) \quad = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\frac{k}{2}+1)} = e^{z^2}\text{erfc}(-z), \tag{3.1.5}$$

where erfc is complementary to the error function efc.

Definition 3.1.5. **Error function.**

$$\text{erfc}(z) : = \frac{2}{\pi^{1/2}} \int_z^{\infty} e^{-u^2} du = 1 - \text{erf}(z), z \in C, \tag{3.1.6}$$

To derive (3.1.5), we see that [[1], p,297, Eq.7.1.] reads as

$$w(z) = e^{-z^2}\text{erfc}(-iz) \tag{3.1.7}$$

whereas [[1],p. 297, Eq.7.1.8] is

$$w(z) = \sum_{n=0}^{\infty} \frac{(iz)^n}{\Gamma(\frac{n}{2}+1)}. \tag{3.1.8}$$

From (3.1.7) and (3.1.8), we easily obtain (3.1.5). In passing, we note that $w(z)$ is also an error function [1].

Definition 3.1.6. **Mellin-Ross function.**

$$E_t(v,a) : = t^v \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(v+k+1)} = t^v E_{1,v+1}(at). \tag{3.1.9}$$

Definition 3.1.7. **Robotov's function.**

$$\text{Ⅎ}_a(\beta,t) : = t^\alpha \sum_{k=0}^{\infty} \frac{\beta^k t^{k(\alpha+1)}}{\Gamma((1+\alpha)(k+1))} = t^\alpha E_{\alpha+1,\alpha+1}(\beta t^{\alpha+1}). \tag{3.1.10}$$

Example 3.1.3. Prove that $E_{1,3}(z) = \frac{e^z - z - 1}{z^2}$.

Solution: We have

$$E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \frac{1}{z^2} \sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+2)!}$$
$$= \frac{1}{z^2}(e^z - z - 1).$$

Example 3.1.4. Prove that

$$E_{1,r}(z) = \frac{1}{z^{r-1}}\left\{e^z - \sum_{k=0}^{r-2} \frac{z^k}{k!}\right\}, r \in \mathbb{N}.$$

The proof is similar to that in Example 3.1.3

3.1. Revision Exercises

3.1.1. Prove that

$$H_{1,2}^{1,1}\left[x\Big|_{(a,A),(0,1)}^{(a,A)}\right] = A^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k x^{(k+a)/A}}{\Gamma[1+(k+a)A]}.$$

3.1.2. Prove that

$$\frac{d}{dx} H_{1,2}^{1,1}\left[x\Big|_{(a,A),(0,1)}^{(a,A)}\right] = H_{1,2}^{1,1}\left[x\Big|_{(a-A,A),(0,1)}^{(a-A,A)}\right].$$

3.1.3. Prove that

$$H_{2,1}^{1,1}\left[\frac{1}{x}\Big|_{(1-a,A)}^{(1-a,A),(1,1)}\right] = A^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{x})^{\frac{k+1-a}{A}}}{\Gamma[1-(k+1-a)/A]}.$$

3.2. Basic Properties of Mittag-Leffler Function

As a consequence of the definitions (3.1.1) and (3.1.2) the following results hold:

Theorem 3.2.1. *There holds the following relations:*

$$\text{(i)}\quad E_{\alpha,\beta}(z) = z\, E_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)} \tag{3.2.1}$$

$$\text{(ii)}\quad E_{\alpha,\beta}(z) = \beta\, E_{\alpha,\beta+1}(z) + \alpha z \frac{\mathrm{d}}{\mathrm{dz}}\, E_{\alpha,\beta+1}(z) \tag{3.2.2}$$

$$\text{(iii)}\quad \left(\frac{\mathrm{d}}{\mathrm{dz}}\right)^m \left[z^{\beta-1}\, E_{\alpha,\beta}(z^\alpha)\right] = z^{\beta-m-1}\, E_{\alpha,\beta-m}(z^\alpha),$$
$$\Re(\beta - m) > 0,\ (m \in \mathbb{N}). \tag{3.2.3}$$

Solutions:

(i) We have

$$\begin{aligned} E_{\alpha,\beta}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \sum_{k=-1}^{\infty} \frac{z^{k+1}}{\Gamma(\alpha + \beta + \alpha k)} \\ &= z\, E_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)},\ \Re(\beta) > 0. \end{aligned}$$

(ii) We have

$$\begin{aligned} R.H.S &= \beta E_{\alpha,\beta+1}(z) + \alpha z \frac{\mathrm{d}}{\mathrm{dz}} \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta + 1)} \\ &= \beta E_{\alpha,\beta+1}(z) + \sum_{k=0}^{\infty} \frac{(\alpha k + \beta - \beta) z^k}{\Gamma(\alpha k + \beta + 1)} \\ &= E_{\alpha,\beta}(z) = L.H.S. \end{aligned}$$

(iii)

$$\begin{aligned} L.H.S &= \left(\frac{\mathrm{d}}{\mathrm{dz}}\right)^m \sum_{k=0}^{\infty} \frac{z^{\alpha k + \beta - 1}}{\Gamma(\alpha k + \beta)} \\ &= \sum_{k=0}^{\infty} \frac{z^{\alpha k + \beta - m - 1}}{\Gamma(\alpha k + \beta - m)},\ \Re(\beta - m) > 0, \end{aligned}$$

since

$$\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{dz}}\right)^m \left(z^{\alpha k + \beta - 1}\right) &= \frac{\Gamma(\alpha k + \beta)}{\Gamma(\alpha k + \beta - m)} z^{\alpha k + \beta - m - 1} \\ &= z^{\beta - m - 1}\, E_{\alpha,\beta-m}(z^\alpha),\quad (m \in \mathbb{N}) \\ &= R.H.S. \end{aligned}$$

Following special cases of (3.2.3) are worth mentioning. If we set $\alpha = \frac{m}{n},\ (m, n \in \mathbb{N})$ then

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m\left[z^{\beta-1}E_{\frac{m}{n},\beta}\left(z^{\frac{m}{n}}\right)\right] = z^{\beta-m-1}E_{\frac{m}{n},\beta-m}\left(z^{\frac{m}{n}}\right)$$
$$= z^{\beta-m-1}\sum_{k=0}^{\infty}\frac{z^{\frac{mk}{n}}}{\Gamma\left(\frac{mk}{n}+\beta-m\right)}$$

for $\Re(\beta - m) > 0$. (Replacing k by $k + n$)

$$= z^{\beta-m-1}\sum_{k=-n}^{\infty}\frac{z^{\frac{mk}{n}}}{\Gamma\left(\beta+\frac{mk}{n}\right)}$$
$$= z^{\beta-1}E_{\frac{m}{n},\beta}\left(z^{\frac{m}{n}}\right) + z^{\beta-1}\sum_{k=1}^{n}\frac{z^{-\frac{mk}{n}}}{\Gamma\left(\beta-\frac{mk}{n}\right)}\ (m, n = 1, 2, 3). \tag{3.2.4}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m\left[z^{\beta-1}E_{m,\beta}\left(z^m\right)\right] = z^{\beta-1}E_{m,\beta}\left(z^m\right) + \frac{z^{-m}}{\Gamma(\beta-m)}\ \text{ for } \Re(\beta-m) > 0. \tag{3.2.5}$$

Putting $z = t^{\frac{n}{m}}$ in (3.2.4) it yields

$$\left(\frac{m}{n}t^{1-\frac{n}{m}}\frac{\mathrm{d}}{\mathrm{d}t}\right)^m\left[t^{(\beta-1)\frac{n}{m}}E_{\frac{m}{n},\beta}(t)\right] = t^{(\beta-1)\frac{n}{m}}E_{\frac{m}{n},\beta}(t) + t^{(\beta-1)\frac{n}{m}}$$
$$\times\sum_{k=1}^{n}\frac{-t^k}{\Gamma\left(\beta-\frac{mk}{n}\right)},\quad Re(\beta-m) > 0, (m, n \in \mathbb{N}). \tag{3.2.6}$$

When $m = 1$ (3.2.6) reduces to

$$\frac{t^{1-n}}{n}\frac{\mathrm{d}}{\mathrm{d}t}\left[t^{(\beta-1)n}E_{\frac{1}{n},\beta}(t)\right] = t^{(\beta-1)n}\ E_{\frac{1}{n},\beta}(t) + t^{(\beta-1)n}\ \sum_{k=1}^{n}\frac{t^{-k}}{\Gamma\left(\beta-\frac{k}{n}\right)},$$

$\Re(\beta) > 1$, which can be written as

$$\frac{1}{n}\ \frac{\mathrm{d}}{\mathrm{d}t}\left[t^{(\beta-1)n}E_{\frac{1}{n},\beta}(t)\right] = t^{\beta n-1}\ E_{\frac{1}{n},\beta}(t) + t^{\beta n-1}\ \sum_{k=1}^{n}\frac{t^{-k}}{\Gamma\left(\beta-\frac{k}{n}\right)}, \Re(\beta) > 1. \tag{3.2.7}$$

3.2.1. Mittag-Leffler functions of rational order

Now we consider the Mittag-Leffler functions of rational order $\alpha = \frac{p}{q}$ with $p, q \in \mathbb{N}$ relatively prime. The following relations readily follow from the definitions (3.1.1) and (3.1.2).

$$\text{(i)} \quad \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^p E_p(z^P) \quad = E_p(z^P) \tag{3.2.8}$$

$$\text{(ii)} \quad \frac{\mathrm{d}^p}{\mathrm{d}z^p} E_{\frac{p}{q}}(z^{\frac{p}{q}}) \quad = E_{\frac{p}{q}}(z^{\frac{p}{q}}) + \sum_{k=1}^{q-1} \frac{z^{-\frac{kp}{q}}}{\Gamma(1-\frac{kp}{q})}; \tag{3.2.9}$$

$q = 1, 2, 3, \cdots$. We now derive the relation

$$\text{(iii)} \quad E_{\frac{1}{q}}\left(z^{\frac{1}{q}}\right) \quad = \mathrm{e}^z\left[1 + \sum_{k=1}^{q-1} \frac{\gamma(1-\frac{k}{q}z)}{\Gamma(1-\frac{k}{q})}\right]; \tag{3.2.10}$$

where $q = 2, 3, \cdots$ and $\gamma(\alpha, z)$ is the incomplete gamma function, defined by

$$\gamma(\alpha, z) = \int_0^z \mathrm{e}^{-u} u^{\alpha-1} \mathrm{d}u.$$

To prove (3.2.10), set $p = 1$ in (3.2.9) and multiply both sides by e^{-z} and use the definition of $\gamma(a, z)$. Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\mathrm{e}^{-z} E_{\frac{1}{q}}\left(z^{\frac{1}{q}}\right)\right] = \mathrm{e}^{-z} \sum_{k=1}^{w-1} \frac{z^{-\frac{k}{q}}}{\Gamma\left(1-\frac{k}{q}\right)}. \tag{3.2.11}$$

Integrating (3.2.11) with respect to z, we obtain (3.2.10).

3.2.2. Euler transform of Mittag-Leffler function

By virtue of beta function formula it is not difficult to show that

$$\int_0^1 z^{\rho-1}(1-z)^{\sigma-1} E_{\alpha,\beta}(xz^{\gamma})\mathrm{d}z = \Gamma(\sigma)\ {}_2\psi_2\left[\begin{matrix}(\rho,\gamma),(1,1)\\(\beta,\alpha),(\sigma+\rho,\gamma)\end{matrix}\middle| x\right] \tag{3.2.12}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\rho) > 0, \Re(\sigma) > 0, \gamma > 0$. Here ${}_2\psi_2$ is the generalized Wright function and $\alpha, \beta, \rho, \sigma \in C$.

Special cases of (3.2.12):

(i) When $\rho = \beta, \gamma = \alpha$, (3.2.12) yields.

$$\int_0^1 z^{\beta-1}(1-z)^{\sigma-1}\, E_{\alpha,\beta}(xz^{\alpha})\mathrm{d}z = \Gamma(\sigma)E_{\alpha,\sigma+\beta}(x), \tag{3.2.13}$$

where $\alpha > 0; \beta, \sigma \in C,\ \Re(\beta) > 0,\ \Re(\sigma) > 0$ and,

(ii)

$$\int_0^1 (z)^{\sigma-1}(1-z)^{\beta-1}\, E_{\alpha,\beta}[x(1-z)^{\alpha}]\mathrm{d}z = \Gamma(\sigma)E_{\alpha,\sigma+\beta}(x), \tag{3.2.14}$$

where $\alpha > 0; \beta, \sigma \in C,\ \Re(\beta) > 0,\ \Re(\sigma) > 0$.

(iii) When $\alpha = \beta = 1$, we have

$$\begin{aligned}\int_0^1 z^{\rho-1}(1-z)^{\sigma-1}\, \exp(xz^{\gamma})\mathrm{d}z &= \Gamma(\sigma)\ {}_2\psi_2\left[{}^{(\rho,\gamma),(1,1)}_{(1,1),(\sigma+\rho,\gamma)}\Big|x\right]\\ &= \Gamma(\sigma)\ {}_1\psi_1\left[{}^{(\rho,\gamma)}_{(\sigma+\rho,\gamma)}\Big|x\right],\end{aligned} \tag{3.2.15}$$

where $\gamma > 0, \rho, \sigma \in C,\ \Re(\rho) > 0,\ \Re(\sigma) > 0$.

3.2.3. Laplace transform of Mittag-Leffler function

By the application of Laplace integral, it follows that

$$\int_0^\infty z^{\rho-1}\mathrm{e}^{-az}\, E_{\alpha,\beta}(xz^{\gamma})\mathrm{d}z = \frac{1}{a^{\rho}}\,{}_2\psi_1\left[{}^{(1,1),(\rho,\gamma)}_{(\beta,\alpha)}\Big|\frac{x}{a^{\gamma}}\right], \tag{3.2.16}$$

where $\rho, a, \alpha, \beta \in C,\ \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(a) > 0,\ \Re(\rho) > 0$ and $\left|\frac{z}{a^{\gamma}}\right| < 1$.

Special cases of (3.2.16) are worth mentioning.

(i) For $\rho = \beta, \gamma = \alpha, \Re(\alpha) > 0$, (3.2.16) gives

$$\int_0^\infty \mathrm{e}^{-az} z^{\beta-1}\, E_{\alpha,\beta}(xz^{\alpha})\mathrm{d}z = \frac{a^{\alpha-\beta}}{a^{\alpha}-x}, \tag{3.2.17}$$

where $a, \alpha, \beta \in C, \Re(\alpha) > 0,\ \Re(\beta) > 0, \Re(a) > 0\ \left|\frac{x}{a^{\alpha}}\right| < 1$.

When $a = 1$, (3.2.17) yields a known result.

$$\int_0^\infty \mathrm{e}^{-z} z^{\beta-1}\, E_{\alpha,\beta}(xz^{\alpha})\mathrm{d}z = \frac{1}{1-x}, |x| < 1, \tag{3.2.18}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0$. If we further take $\beta = 1$, (3.2.18) reduces to

$$\int_0^\infty e^{-z}\ E_\alpha(xz^\alpha)dz = \frac{1}{1-x}, |x| < 1.$$

(ii) When $\beta = 1$, (3.2.17) gives

$$\int_0^\infty e^{-az}\ E_\alpha(xz^\alpha)dz = \frac{a^{\alpha-1}}{a^\alpha - x}, \tag{3.2.19}$$

where $\Re(a) > 0, \Re(\alpha) > 0,\ \left|\frac{x}{a^\alpha}\right| < 1.$

3.2.4. Application of Laplace transform

From (3.2.17), we find that

$$L\left\{x^{\beta-1}E_{\alpha,\beta}(ax^\alpha)\right\} = \frac{s^{\alpha-\beta}}{s^\alpha - a} \tag{3.2.20}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0$

$$L\{f(t); s\} = \int_0^\infty e^{-st} f(t)dt, \Re(s) > 0. \tag{3.2.21}$$

We also have

$$L\left\{x^{\gamma-1}E_{\alpha,\gamma}(-ax^\alpha)\right\} = \frac{s^{\alpha-\gamma}}{s^\alpha + a} \tag{3.2.22}$$

Now

$$\left[\frac{s^{\alpha-\beta}}{s^\alpha - a}\right]\left[\frac{s^{\alpha-\gamma}}{s^\alpha + a}\right] = \frac{s^{2\alpha-(\beta+\gamma)}}{s^{2\alpha} - a^2}\ \text{ for } \Re(s^2) > \Re(a). \tag{3.2.23}$$

By virtue of the convolution theorem of the Laplace transform, it readily follows that

$$\int_0^t u^{\beta-1}E_{\alpha,\beta}(au^\alpha)(t-u)^{\gamma-1}\ E_{\alpha,\gamma}(-a(t-u)^\alpha)du$$
$$= t^{\beta+\gamma-1}\ E_{2\alpha,\beta+\gamma}(a^2t^{2\alpha}). \tag{3.2.24}$$

where $\Re(\beta) > 0, \Re(\gamma) > 0$. Further, if we use the identity

$$\frac{1}{s^2} = \frac{s^{\alpha-\beta}}{s^\alpha - 1}\ \left[s^{\beta-2} - s^{\beta-\alpha-2}\right] \tag{3.2.25}$$

and the relation

$$L\left\{t^{\rho-1}; s\right\} = \Gamma(\rho)s^{-\rho}, \tag{3.2.26}$$

where $\Re(\rho) > 0$, $\Re(s) > 0$, we obtain

$$\int_0^t u^{\beta-1}\, E_{\alpha,\beta}(u^\alpha)\left[\frac{(t-u)^{1-\beta}}{\Gamma(2-\beta)} - \frac{(t-u)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right]\mathrm{d}u = t, \tag{3.2.27}$$

where $0 < \beta < 2, \Re(\alpha) > 0$.

Next we note that the following result (3.2.29) can be derived by the application of Laplace transform of the identity

$$\left[\frac{s^{2\alpha-\beta}}{s^{2\alpha}-1}\right][s^{-\alpha}] = -\frac{s^{2\alpha-\beta}}{s^{2\alpha}-1} + \frac{s^{\alpha-\beta}}{s^{\alpha}-1} \ \text{ for } \Re(s^\alpha) > 1. \tag{3.2.28}$$

We have

$$\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\, E_{2\alpha,\beta}(t^{2\alpha})t^{\beta-1}\mathrm{d}t$$
$$= -x^{\beta-1}\, E_{2\alpha,\beta}(x^{2\alpha}) + x^{\beta-1}\, E_{\alpha,\beta}(x^\alpha), \tag{3.2.29}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0$. If we set $\beta = 1$ in (3.2.29), it reduces to

$$\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\, E_{2\alpha}(t^{2\alpha})\,\mathrm{d}t = E_\alpha(x^\alpha) - E_{2\alpha}(x^{2\alpha}) \tag{3.2.30}$$

where $\Re(\alpha) > 0$.

3.2.5. Contour integral representation

Lemma 3.2.1. *Contour integral representation for $E_\alpha(z)$ is given by*

$$E_\alpha(z) = \frac{1}{2\pi i}\int_{Ha} \frac{t^{\alpha-1}\mathrm{e}^t\mathrm{d}t}{t^\alpha - z}, \tag{3.2.31}$$

where the path of integration Ha is a loop starting and ending at $-\infty$, and encircling the circular disk $|t| \le |z|^{1/\alpha}$ in the positive sense : $-\pi < \arg t \le \pi$ on Ha.

To establish the representation (3.2.31) we expand the integrand in powers of z, integrate term by term and apply the Hankel integral for the reciprocal of the gamma function, namely

$$\int_{Ha} \mathrm{e}^s s^{-z}\mathrm{d}s = \frac{2\pi i}{\Gamma(z)}. \tag{3.2.32}$$

Similarly we can establish

Lemma 3.2.2.

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_{Ha} \frac{t^{\alpha-\beta}e^t dt}{t^\alpha - z}. \tag{3.2.33}$$

where $\alpha, \beta > 0$.

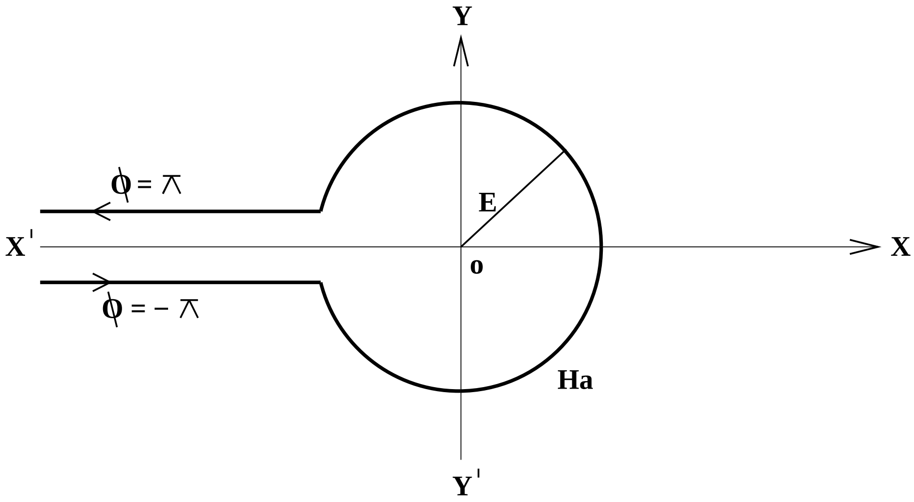

Figure 3.1: **Showing Hankel contour Ha**

3.2.6. Relation between Mittag-Leffler functions and the H-function

Both the Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ belong to H-function family. We derive their relations with the H function.

Lemma 3.2.3. *Let* $\alpha \in \mathbb{R} = (0, \infty)$*. Then* $E_\alpha(z)$ *is represented by the Mellin-Barnes integral*

$$E_\alpha(z) = \frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(1-s)(-z)^{-s}}{\Gamma(1-\alpha s)} ds, \quad (|\arg z| < \pi), \tag{3.2.34}$$

where the contour of integration L*, beginning at* $-i\infty$ *and ending at* $+i\infty$*, separates all poles* $s = -k$ $(k \in \mathbb{N}_0)$ *to the left and all poles* $s = 1 + n$ $(n \in \mathbb{N}_0)$ *to the right.*

Proof. We now evaluate the integral (3.2.34) as the sum of the residues at the points $s = 0, -1, -2, \cdots$. We find that

$$\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(1-s)(-z)^{-s}}{\Gamma(1-\alpha s)}\mathrm{d}s = \sum_{k=0}^{\infty} \mathrm{Res}_{s=-k}\Big[\frac{\Gamma(s)\Gamma(1-s)(-z)^{-s}}{\Gamma(1-\alpha s)}\Big] \tag{3.2.35}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k\Gamma(1+k)}{k!\Gamma(1+\alpha k)}(-z)^k$$
$$= E_\alpha(z),$$

which yields (3.2.34) in accordance with the definition (3.1.1). It readily follows from the definition of the H - function and (3.2.34) that $E_\alpha(z)$ can be represented in the form

$$E_\alpha(z) = H^{1,1}_{1,2}\Big[-z\Big|^{(0,1)}_{(0,1),\ (0,\alpha)}\Big], \tag{3.2.36}$$

where $H^{1,1}_{1,2}$ is the H-function, which is studied in Chapter 1.

Lemma 3.2.4. *Let $\alpha \in \mathbb{R}_+ = (0,\infty), \beta \in C$, then*

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(1-s)(-z)^{-s}}{\Gamma(\beta-\alpha s)}\mathrm{d}s. \tag{3.2.37}$$

The proof of (3.2.37) is similar to that of (3.2.34). Hence the proof is omitted. From (3.2.37) and the definition of the H-function we obtain the relation

$$E_{\alpha,\beta}(z) = H^{1,1}_{1,2}\Big[-z\Big|^{(0,1)}_{(0,1),\ (\beta,\alpha)}\Big]. \tag{3.2.38}$$

In particular, $E_\alpha(z)$ can be expressed in terms of generalized Wright function in the form

$$E_\alpha(z) = {}_1\psi_1\Big[{}^{(1,1)}_{(1,\alpha)}\Big|z\Big]. \tag{3.2.39}$$

Similarly, we have

$$E_\alpha,\beta(z) = {}_1\psi_1\Big[{}^{(1,1)}_{(\beta,\alpha)}\Big|z\Big]. \tag{3.2.40}$$

Next if we calculate the residues at the poles of the gamma function $\Gamma(1-s)$ at the points $s = 1+n$ $(n = 0,1,2,\cdots)$ it gives

$$\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)}(-z)^s \mathrm{d}s = \sum_{n=0}^{\infty} \mathrm{Res}_{s\to 1+n}\Big[\frac{\Gamma(s)\Gamma(1-s)(-z)^{-s}}{\Gamma(1-\alpha s)}\Big]$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n1}\frac{\Gamma(1+n)(-z)^{-n-1}}{\Gamma(1-\alpha(1+n))}$$
$$= -\sum_{n=1}^{\infty} \frac{z^{-n}}{\Gamma(1-\alpha n)}. \tag{3.2.41}$$

Similarly for $E_\alpha,\beta(z)$, if we calculate the residues at $s = 1 + n$ $(n = 0, 1, 2, \cdots)$, we obtain

$$\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}\mathrm{d}s = -\sum_{n=1}^{\infty}\frac{z^{-n}}{\Gamma(\beta-\alpha n)} \tag{3.2.42}$$

Exercises 3.2.

3.2.1. Let

$$U_1(t) = t^{\beta-1}E_{\frac{m}{n},\beta}(t^{\frac{m}{n}})$$
$$U_2(t) = t^{\beta-1}E_{m,\beta}(t^m)$$
$$U_3(t) = t^{(\beta-1)\frac{n}{m}}E_{\frac{m}{n},\beta}(t)$$

and

$$U_4(t) = t^{(\beta-1)n}E_{\frac{1}{n},\beta}(t).$$

Then show that these functions respectively satisfy the following differential equations of Mittag-Leffler functions.

(i) $\dfrac{\mathrm{d}^m}{\mathrm{d}t^m}U_1(t) - U_1(t) = t^{\beta-1}\sum_{k=1}^{n}\dfrac{t^{-\frac{m}{n}k}}{\Gamma(\beta-\frac{mk}{n})}$

$\Re(\beta) > m, (m, n = 1, 2, 3, \cdots)$

(ii) $\dfrac{\mathrm{d}^m}{\mathrm{d}t^m}U_2(t) - U_2(t) = \dfrac{t^{-m+\beta-1}}{\Gamma(\beta-m)}, \Re(\beta) > m, (m = 1, 2, 3, \cdots)$

(iii) $\left(\frac{m}{n}t^{1-\frac{n}{m}}\frac{\mathrm{d}}{\mathrm{d}t}\right)^m U_3(t) - U_3(t) = t^{(\beta-1)\frac{n}{m}} \sum_{k=1}^{n} \frac{t^{-k}}{\Gamma(\beta - \frac{mk}{n})}$
$(m, n = 1, 2, 3, \cdots)$

(iv) $\frac{1}{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}(U_4(t))\right] - t^{n-1}U_4(t) = t^{\beta n-1} \sum_{k=1}^{n} \frac{t^{-k}}{\Gamma(\beta - \frac{k}{n})}$
$(n = 1, 2, 3, \cdots)$.

3.2.2. Prove that

$$\frac{\lambda}{\Gamma(\alpha)}\int_0^x \frac{E_\alpha(\lambda t^\alpha)\mathrm{d}t}{(x-t)^{1-\alpha}} = E_\alpha(\lambda x^\alpha) - 1, \quad \Re(\alpha) > 0.$$

3.2.3. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\gamma-1}E_{\alpha,\beta}(ax^\alpha)\right] = x^{\gamma-2}E_{\alpha,\beta-1}(ax^\alpha) + (\gamma-\beta)x^{\gamma-2}E_{\alpha,\beta}(ax^\alpha), \quad \beta \neq \gamma.$$

3.2.4. Prove that

$$\frac{1}{\Gamma(\upsilon)}\int_0^z t^{\beta-1}(z-t)^{\upsilon-1}E_{\alpha,\beta}(\lambda t^\alpha)\mathrm{d}t = z^{\beta+\upsilon-1}E_{\alpha,\beta+\upsilon}(\lambda z^\alpha), \quad \Re(\beta) > 0, \quad \Re(\upsilon) > 0 \ \Re(\alpha) > 0.$$

3.2.5. Prove that

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}\cosh(\sqrt{\lambda}\, t)\mathrm{d}t = z^\alpha E_{2,\alpha+1}(\lambda z^2), \quad \Re(\alpha) > 0.$$

3.2.6. Prove that

$$\frac{1}{\Gamma(\alpha)}\int_0^z \mathrm{e}^{\lambda t}(z-t)^{\alpha-1}\mathrm{d}t = z^\alpha E_{1,\alpha+1}(\lambda z), \ \Re(\alpha) > 0.$$

3.2.7. Prove that

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}\frac{\sinh\left(\sqrt{\lambda}\, t\right)}{\sqrt{\lambda}}\mathrm{d}t = z^{\alpha+1}E_{2,\alpha+2}(\lambda z^2), \quad \Re(\alpha) > 0.$$

3.2.8. Prove that

$$\int_0^\infty e^{-s\rho}\rho^{\beta-1}\, E_{\alpha,\beta}(\rho^\alpha)d\rho = \frac{s^{\alpha-\beta}}{s^\alpha - 1},\ \Re(s) > 1.$$

3.2.9. Prove that

$$\int_0^\infty e^{-st}E_\alpha(t^\alpha)dt = \frac{1}{s - s^{1-\alpha}},\ \Re(s) > 1.$$

3.2.10. Prove that

$$\int_0^x u^{\gamma-1}E_{\alpha,\gamma}(yu^\alpha)(x-u)^{\beta-1}E_{\alpha,\beta}[z(x-u)^\alpha]du$$
$$= \frac{yE_{\alpha,\beta+\gamma}(yx^\alpha) - zE_{\alpha,\beta+\gamma}(zx^\alpha)}{y-z}x^{\beta+\gamma-1}$$

where $y, z \in C; y \neq z,\ \ \gamma > 0, \beta > 0.$

3.3. Generalized Mittag-Leffler Function

Notation 3.3.1. $E^\delta_{\beta,\gamma}(z)$: **Generalized Mittag-Leffler function**

Definition 3.3.1.

$$E^\delta_{\beta,\gamma}(z) := \sum_{n=0}^\infty \frac{(\delta)_n\, z^n}{\Gamma(\beta n + \gamma)n!}, \tag{3.3.1}$$

where $\beta, \gamma, \delta \in C$ with $\Re(\beta) > 0$. For $\delta = 1$, it reduces to Mittag-Leffler function (3.1.2). This function was introduced by T. R. Prabhakar in 1971. It is an entire function of order $\rho = [\Re(\beta)]^{-1}$.

3.3.1. Special cases of $E^\delta_{\beta,\gamma}(z)$

$$\text{(i)}\ \ E_\beta(z) = E^1_{\beta,1}(z). \tag{3.3.2}$$

$$\text{(ii)}\ \ E_\beta, \gamma(z) = E^1_{\beta,\gamma}(z) \tag{3.3.3}$$

$$\text{(iii)}\ \ \phi(\gamma, \delta; z) = {}_1F_1(\gamma; \delta; z) = \Gamma(\delta)E^\gamma_{1,\delta}(z), \tag{3.3.4}$$

where $\phi(\gamma, \delta; z)$ is Kummer's confluent hypergeometric function.

3.3.2. Mellin-Barnes integral representation

Lemma 3.3.1. *Let $\beta \in \mathbb{R}_+ = (0, \infty); \gamma, \delta \in C(\gamma \neq 0)$. Then $E^{\delta}_{\beta,\gamma}(z)$ is represented by the Mellin-Barnes integral*

$$E^{\delta}_{\beta,\gamma}(z) = \frac{1}{2\pi i \ \Gamma(\delta)} \int_L \frac{\Gamma(s)\Gamma(\delta - s)}{\Gamma(\gamma - \beta s)} (-z)^{-s} \mathrm{d}s, \tag{3.3.5}$$

where $|\arg(z)| < \pi$; *the contour of integration beginning at* $-i\infty$ *and ending at* $+i\infty$, *separates all the poles at* $s = -k$ $(k \in \mathbb{N}_0)$ *to the left and all the poles at* $s = n + \delta$ $(n \in \mathbb{N}_0)$ *to the right.*

Proof. We will evaluate the integral on the R.H.S of (3.3.5) as the sum of the residues at the poles $s = 0, -1, -2, \cdots$. We have

$$\begin{aligned}\frac{1}{2\pi i} \int_L \frac{\Gamma(s)\Gamma(\delta - s)}{\Gamma(\gamma - \beta s)} (-z)^{-s} \mathrm{d}s &= \sum_{k=0}^{\infty} \mathrm{Res}_{s=-k} \Big[\frac{\Gamma(s)\Gamma(\delta - s)}{\Gamma(\gamma - \beta s)} (-z)^{-s} \Big] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(\delta + k)}{\Gamma(\gamma + \beta k)} (-z)^k \\ &= \Gamma(\delta) \sum_{k=0}^{\infty} \frac{(\delta)_k}{\Gamma(\beta k + \gamma)} \frac{z^k}{k!} = \Gamma(\delta) E^{\delta}_{\beta,\gamma}(z)\end{aligned}$$

which proves (3.3.5).

3.3.3. Relations with the H function and Wright hypergeometric function

It follows from (3.3.5) that $E^{\delta}_{\beta,\gamma}(z)$ can be represented in the form

$$E^{\delta}_{\beta,\gamma}(z) = \frac{1}{\Gamma(\delta)} H^{1,1}_{1,2} \Big[-z \Big|^{(1-\delta,1)}_{(0,1),(1-\gamma,\beta)} \Big] \tag{3.3.6}$$

where $H^{1,1}_{1,2}(z)$ is the H function, the theory of which can be found in Chapter 1. This function can also be represented by

$$E^{\delta}_{\beta,\gamma}(z) = \frac{1}{\Gamma(\delta)} {}_1\psi_1 \Big[{}^{(\delta,1)}_{(\gamma,\beta)} \Big| z \Big], \tag{3.3.7}$$

where ${}_1\psi_1$ is the Wright hypergeometric function ${}_p\psi_q(z)$.

3.3.4. Cases of reducibility

In this subsection, we present some interesting cases of reducibility of the function $E^{\delta}_{\beta,\gamma}(z)$. The results are given in the form of five theorems. The results are useful in the investigation of the solutions of certain fractional-order differential and integral equations. The proof of the following theorems can be developed on similar lines to that of equation (3.2.1).

Theorem 3.3.1. *If $\beta, \gamma, \delta \in C$ with $\Re(\beta) > 0, \Re(\gamma) > 0, \Re(\gamma - \beta) > 0$, then there holds the relation*

$$zE^{\delta}_{\beta,\gamma}(z) = E^{\delta}_{\beta,\gamma-\beta}(z) - E^{\delta-1}_{\beta,\gamma-\beta}(z). \tag{3.3.8}$$

Corollary 3.3.1. *If $\beta, \gamma, \delta \in C$, $\Re(\gamma) > \Re(\beta) > 0$, then we have*

$$z\, E^{\delta}_{\beta,\gamma}(z) = E_{\beta,\gamma-\beta}(z) - \frac{1}{\Gamma(\gamma-\beta)}. \tag{3.3.9}$$

Theorem 3.3.2. *If $\beta, \gamma, \delta \in C$, $\Re(\beta) > 0, \Re(\gamma) > 1$, then there holds the formula*

$$\beta\, E^{2}_{\beta,\gamma}(z) = E_{\beta,\gamma-1}(z) + (1+\beta-\gamma)E_{\beta,\gamma}(z). \tag{3.3.10}$$

Theorem 3.3.3. *If $\Re(\beta) > 0, \Re(\gamma) > 2 + \Re(\beta)$, then there holds the formula*

$$\begin{aligned} zE^{3}_{\beta,\gamma}(z) = \frac{1}{2\beta^2}\Big[&E_{\beta,\gamma-\beta-2}(z) - (2\gamma - 3\beta - 3)E_{\beta,\gamma-\beta-1}(z) \\ &+ (2\beta^2 + \gamma^2 - 3\beta\gamma + 3\beta\gamma - 2\gamma + 1)\ E_{\beta,\gamma-\beta}(z)\Big]. \end{aligned} \tag{3.3.11}$$

Theorem 3.3.4. *If $\Re(\beta) > 0, \Re(\gamma) > 2$, then there holds the formula*

$$\begin{aligned} E^{3}_{\beta,\gamma}(z) = \frac{1}{2\beta^2}\Big[&E_{\beta,\gamma-2}(z) + (3 + 3\beta - 2\gamma)E_{\beta,\gamma-1}(z) \\ &+ (2\beta^2 + \gamma^2 + 3\beta - 3\beta\gamma - 2\gamma + 1)E_{\beta,\gamma}(z)\Big]. \end{aligned} \tag{3.3.12}$$

3.3.5. Differentiation of generalized Mittag-Leffler function

Theorem 3.3.5. *Let $\beta, \gamma, \delta, \rho, w \in \mathbb{C}$. Then for any $n \in \mathbb{N}$, there holds the formula, for $\Re(\gamma) > n$,*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \left[z^{\gamma-1} E^{\delta}_{\beta,\gamma}(wz^{\beta})\right] = z^{\gamma-n-1} E^{\delta}_{\beta,\gamma-n}(wz^{\beta}). \tag{3.3.13}$$

In particular for $\Re(\gamma) > n$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \left[z^{\gamma-1} E_{\beta,\gamma}(wz^{\beta})\right] = z^{\gamma-n-1} E_{\beta,\gamma-n}(wz^{\beta}) \tag{3.3.14}$$

and for $\Re(\gamma) > n$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \left[z^{\gamma-1} \phi(\delta;\gamma;wz)\right] = \frac{\Gamma(\gamma)}{\Gamma(\gamma-n)} z^{\gamma-n-1} \phi(\delta,\gamma-n;wz). \tag{3.3.15}$$

Proof. Using (3.3.1) and taking term by term differentiation under the summation sign, which is possible in accordance with uniform convergence of the series in (3.3.1) in any compact set of $\mathbb{C}$, we obtain

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \left[z^{\gamma-1} E^{\delta}_{\beta,\gamma}(wz^{\beta})\right] &= \sum_{k=0}^{\infty} \frac{(\delta)_k}{\Gamma(\beta k+\gamma)} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \left[\frac{w^k z^{\beta\gamma+\gamma-1}}{k!}\right] \\ &= z^{\gamma-n-1} E^{\delta}_{\beta,\gamma-n}(wz^{\beta}) \text{ for } \Re(\gamma) > n,\end{aligned}$$

which establishes (3.3.13). Note that (3.3.14) follows from (3.3.13) when $\delta = 1$ due to (3.3.3) and (3.3.15) follows from (3.3.13) when $\beta = 1$ on account of (3.3.4).

3.3.6. Integral property of generalized Mittag-Leffler function

Corollary 3.3.2. *Let $\beta, \gamma, \delta, w \in \mathbb{C}$, $\Re(\gamma) > 0, \Re(\beta) > 0, \Re(\delta) > 0$. Then*

$$\int_0^z t^{\gamma-1} E^{\delta}_{\beta,\gamma}(wt^{\beta})\mathrm{d}t = z^{\gamma} E^{\delta}_{\beta,\gamma+1}(wz^{\beta}) \tag{3.3.16}$$

and (3.3.16) follows from (3.3.13). In particular,

$$\int_0^z t^{\gamma-1} E_{\beta,\gamma}(wt^{\beta})\mathrm{d}t = z^{\gamma} E_{\beta,\gamma+1}(wt^{\beta}) \tag{3.3.17}$$

and

$$\int_0^z t^{\gamma-1} \phi(\gamma,\delta;wt)\mathrm{d}t = \frac{1}{\delta} z^{\delta} \phi(\gamma,\delta+1;wx) \tag{3.3.18}$$

Remark 3.3.1. The relations (3.3.15) and (3.3.18) are well known.

3.3.7. Integral transform of $E^{\delta}_{\beta,\gamma}(z)$

By appealing to the Mellin inversion formula (3.3.5) yields the Mellin-transform of the generalized Mittag-Leffler function.

$$\int_0^\infty t^{s-1}\, E^{\delta}_{\beta,\gamma}(-wt)\mathrm{d}t = \frac{\Gamma(s)\Gamma(\delta - s)}{\Gamma(\delta)\, w^s\Gamma(\gamma - s\beta)}. \tag{3.3.19}$$

If we make use of the integral

$$\int_0^\infty t^{\upsilon-1}\mathrm{e}^{-\frac{t}{2}} W_{\lambda,\mu}(t)\mathrm{d}t = \frac{\Gamma(\frac{1}{2} + \mu + \upsilon)\Gamma(\frac{1}{2} - \mu + \upsilon)}{\Gamma(1 - \lambda + \upsilon)} \tag{3.3.20}$$

where $\Re(\upsilon \pm \mu) > -\frac{1}{2}$, we obtain the Whittaker transform of the Mittag-Leffler function

$$\int_0^\infty t^{\rho-1}\mathrm{e}^{-\frac{1}{2}pt} W_{\lambda,\mu}(pt) E^{\delta}_{\beta,\gamma}(wt^{\alpha})\mathrm{d}t = \frac{p^{-\rho}}{\Gamma(\delta)}\, {}_3\psi_2\left[{}^{(\delta,1),(\frac{1}{2}\pm\mu+\rho,\alpha)}_{(\gamma,\beta),(1-\lambda+\rho,\alpha)} \Big| \frac{w}{p^{\alpha}} \right] \tag{3.3.21}$$

where ${}_3\psi_2(.)$ is the generalized Wright function, and $\Re(\rho) > |\Re(\mu)| - \frac{1}{2},\ \Re(p) > o, |\frac{w}{p^{\alpha}}| < 1$. When $\lambda = 0$ and $\mu = \frac{1}{2}$, then by virtue of the identity

$$W_{\pm\frac{1}{2},0}(t) = \exp(-\frac{t}{2}), \tag{3.3.22}$$

the Laplace transform of the generalized Mittag-Leffler function is obtained:

$$\int_0^\infty t^{\rho-1}\mathrm{e}^{-pt} E^{\delta}_{\beta,\gamma}(wt^{\alpha})\mathrm{d}t = \frac{p^{-\rho}}{\Gamma(\delta)}\, {}_2\psi_1\left[{}^{(\delta,1),(\rho,\alpha)}_{(\gamma,\beta)} \Big| \frac{w}{p^{\alpha}} \right] \tag{3.3.23}$$

where $\Re(\beta) > 0, \Re(\gamma) > 0, \Re(\rho) > 0,\ \Re(p) > 0, p > |w|^{\frac{1}{\Re(\alpha)}}$. In particular, for $\rho = \gamma$ and $\alpha = \beta$ we obtain a result given by Prabhakar [p.8, Eq. 2.5].

$$\int_0^\infty t^{\gamma-1}\mathrm{e}^{-pt} E^{\delta}_{\beta,\gamma}(wt^{\beta})\mathrm{d}t = p^{-\gamma}(1 - wp^{-\beta})^{-\delta} \tag{3.3.24}$$

where $\Re(\beta) > 0, \Re(\gamma) > 0,\ \Re(p) > 0$ and $p > |w|^{\frac{1}{\Re(\beta)}}$.

The Euler transform of the generalized Mittag-Leffler function follows from the beta function:

$$\int_0^1 t^{a-1}(1 - t)^{b-1}\, E^{\delta}_{\beta,\gamma}(xt^{\alpha})\mathrm{d}t = \frac{\Gamma(b)}{\Gamma(\delta)}\, {}_2\psi_2\left[{}^{(\delta,1),(a,\alpha)}_{(\gamma,\beta),(a+b,\alpha)} \Big| x \right], \tag{3.3.25}$$

where $\Re(a) > 0,\ \Re(b) > 0, \Re(\delta) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \alpha > 0$.

Theorem 3.3.6. *We have*

$$\int_0^\infty e^{-pt} t^{\alpha k+\beta-1} \, E^{(k)}_{\alpha,\beta}(\pm a t^\alpha) dt, = \frac{k! p^{\alpha-\beta}}{(p^\alpha \mp a)^{k+1}}, \tag{3.3.26}$$

where $\Re(p) > |a|^{1/\alpha}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, *and* $E^{(k)}_{\alpha,\beta}(y) = \frac{d^k}{dy^k} E_{\alpha,\beta}(y)$.

Solution: We will use the following result:

$$\int_0^\infty e^{-t} t^{\beta-1} \, E_{\alpha,\beta}(z t^\alpha) dt = \frac{1}{1-z} \quad (|z| < 1). \tag{3.3.27}$$

The given integral

$$= \frac{d^k}{da^k} \int_0^\infty e^{-pt} t^{\beta-1} E_{\alpha,\beta}(\pm a t^\alpha) dt$$
$$= \frac{d^k}{da^k} \frac{p^\alpha - \beta}{(p^\alpha \mp a)} = \frac{k! p^{\alpha-\beta}}{(p^\alpha \mp a)^{k+1}}, \quad \Re(\beta) > 0.$$

Corollary 3.3.3.

$$\int_0^\infty e^{-pt} t^{\frac{k-1}{2}} \, E^{(k)}_{\frac{1}{2},\frac{1}{2}}(\pm a \sqrt{t}) dt = \frac{k!}{(\sqrt{p} \mp a)^{k+1}} \tag{3.3.28}$$

where $\Re(p) > a^2$.

Exercises 3.3.

3.3.1. prove that

$$\frac{1}{\Gamma(\alpha)} \int_0^1 u^{\gamma-1} (1-u)^{\alpha-1} E^\delta_{\beta,\gamma}(z u^\beta) du = E^\delta_{\beta,\gamma+\alpha}(z), \ \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0.$$

3.3.2. Prove that

$$\frac{1}{\Gamma(\alpha)} \int_t^x (x-s)^{\alpha-1} (s-t)^{\gamma-1} E^\delta_{\beta,\gamma}[\lambda (s-t)^\beta] ds = (x-t)^{\gamma+\alpha-1} \, E^\delta_{\beta,\gamma+\alpha}[\lambda (x-t)^\beta]$$

where $\Re(\alpha) > 0, \Re(\gamma) > 0$.

3.3.3. Prove that

$$\frac{1}{\Gamma(\alpha)}\int_t^x (s-t)^{\alpha-1}(x-s)^{\gamma-1}E_{\beta,\gamma}^{\delta}[\lambda(x-s)^{\beta}]\mathrm{d}s = (x-t)^{\gamma+\alpha-1}\, E_{\beta,\gamma+\alpha}^{\delta}[\lambda(x-t)^{\beta}]$$

where $\Re(\alpha) > 0, \Re(\gamma) > 0, \Re(\beta) > 0$.

3.3.4. Prove that for $n = 1, 2, \cdots$

$$E_{n,\gamma}^{\delta}(z) = \frac{\pi^{\frac{n-1}{2}} n^{\frac{1}{2}-\gamma}}{\Gamma(\gamma)}\, {}_1F_n(\delta; \Delta(n;\gamma); n^n z),$$

where $\Delta(n;\gamma)$ represents the sequence of n parameters $\frac{\gamma}{n}, \frac{r+1}{n}, \frac{\gamma+n-1}{n}$.

3.3.5. Show that for $\Re(\beta) > 0, \quad \Re(\gamma) > 0$,

$$\left(\frac{\mathrm{d}}{\mathrm{dz}}\right)^m E_{\beta,\gamma}^{\delta}(z) = (\delta)_m E_{\beta,\gamma+m\beta}^{\delta+m}(z).$$

3.3.6. Prove that for $\Re(\beta) > 0, \quad \Re(\gamma) > 0$,

$$\left(z\frac{\mathrm{d}}{\mathrm{dz}} + \delta\right) E_{\beta,\gamma}^{\delta}(z) = \delta\, E_{\beta,\gamma}^{\delta+1}(z).$$

3.3.7. Prove that for $\Re(\gamma) > 1$,

$$(\gamma - \beta\delta - 1)\, E_{\beta,\gamma}^{\delta}(z) = E_{\beta,\gamma-1}^{\delta}(z) - \beta\delta E_{\beta,\gamma}^{\delta+1}(z).$$

3.3.8. Prove that

$$\int_0^x t^{\upsilon-1}(x-t)^{\mu-1}\, E_{\rho,\mu}^{\gamma}(w[x-t]^{\rho})\, E_{\rho,\upsilon}^{\sigma}(wt^{\rho})\mathrm{d}t = x^{\mu+\upsilon-1}E_{\rho,\mu+\upsilon}^{\gamma+\sigma}(wx^{\rho}),$$

where $\rho, \mu, \gamma, \upsilon, \sigma, w \in \mathbb{C}$; $\Re(\rho), \Re(\mu), \Re(\upsilon) > 0$.

3.3.9. Find

$$L^{-1}\left\{s^{-\lambda}(1-\frac{z}{s^{\rho}})^{-\alpha}\right\}$$

and give the conditions of validity.

3.3.10. Prove that

$$L^{-1}\left[s^{-\lambda}(1-\frac{z_1}{s})^{-\alpha_1}(1-\frac{z_2}{s})^{-\alpha_2}\right]=\frac{t^{\lambda-1}}{\Gamma(\lambda)}\,\Phi_2[\alpha_1,\alpha_2;\lambda;z_1t,z_2t],$$

where $\Re(\lambda) > 0, \Re(s) > \max[0, \Re(z_1), \Re(z_2)]$ and $\Phi_2(.)$ is the confluent hypergeometric function of two variables defined by

$$\Phi_2(b,b';c;u,z)=\sum_{k,l=0}^{\infty}\frac{(b)_k(b')_l u^k z^l}{(c)_{k+l}k!l!}.$$

3.3.11. From the above result deduce the formula

$$L^{-1}\left\{s^{-\lambda}(1-z/s)^{-\alpha}\right\}=\frac{t^{\lambda-1}}{\Gamma(\lambda)}\,\phi(\alpha,\lambda;zt),$$

where $\Re(\lambda) > 0, \Re(s) > \max[0, |z|]$.

3.4. Fractional Integrals

This section deals with the definition and properties of various operators of fractional integration and fractional differentiation of arbitrary order. Among the various operators studied, it involves the Riemann-Liouville fractional integral operators, Riemann-Liouville fractional differentiation operators, Weyl operators and Kober operators etc. Besides the basic properties of these operators, their behaviours under Laplace, Fourier and Mellin transforms are also presented. Application of Riemann-Liouville operators in the solution of fractional order differential and fractional order integral equations is demonstrated.

3.4.1. Riemann-Liouville fractional integrals of arbitrary order

Notation 3.4.1. ${}_aI_x^n, {}_aD_x^{-n}, n \in \mathbb{N}\cup\{0\}$: **Fractional integral of integer order** n

Definition 3.4.1.

$$ {}_aI_x^n f(x) = {}_aD_x^{-n} f(x) = \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f(t)\mathrm{d}t \tag{3.4.1}$$

where $n \in \mathbb{N}\cup\{0\}$.

We begin our study of fractional calculus by introducing a fractional integral of integer order n in the form (Cauchy formula):

$$ {}_aD_x^{-n} f(x) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)\mathrm{d}t. \tag{3.4.2} $$

It will be shown that the above integral can be expressed in terms of n-fold integral, that is,

$$ {}_aD_x^{-n} f(x) = \int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \int_a^{x_2} \mathrm{d}x_3 \cdots \int_a^{x_{n-1}} f(t)\mathrm{d}t. \tag{3.4.3} $$

Proof. When $n = 2$, then using the well-known Dirichlet formula, namely

$$ \int_a^b \mathrm{d}x \int_a^x f(x,y)\mathrm{d}y = \int_a^b \mathrm{d}y \int_y^b f(x,y)\mathrm{d}x \tag{3.4.4} $$

(3.4.3) becomes

$$ \begin{aligned} \int_a^x \mathrm{d}x_1 \int_a^{x_1} f(t)\mathrm{d}t &= \int_a^x \mathrm{d}t\ f(t) \int_t^x \mathrm{d}x_1 \\ &= \int_a^x (x-t) f(t)\mathrm{d}t. \end{aligned} \tag{3.4.5} $$

This shows that the two-fold integral can be reduced to a single integral with the help of Dirichlet formula. For $n = 3$, the integral in (3.4.3) gives

$$ \begin{aligned} {}_aD_x^{-3} f(x) &= \int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \int_a^{x_2} f(t)\mathrm{d}t \\ &= \int_a^x \mathrm{d}x_1 \Big[\int_a^{x_1} \mathrm{d}x_2 \int_a^{x_2} f(t)\mathrm{d}t \Big]. \end{aligned} \tag{3.4.6} $$

Using the result (3.4.5) the integrals within big brackets simplify to yield

$$ {}_aD_x^{-3} f(x) = \int_a^x \mathrm{d}x_1 \Big[\int_a^{x_1} (x_1 - t) f(t)\mathrm{d}t \Big]. \tag{3.4.7} $$

If we use (3.4.4), then the above expression reduces to

$$ {}_aD_x^{-3} f(x) = \int_a^x \mathrm{d}t\, f(t) \int_x^t (x_1 - t)\mathrm{d}x_1 = \int_a^x \frac{(x-t)^2}{2!} f(t)\mathrm{d}t. \tag{3.4.8} $$

Continuing this process, we finally obtain

$$ {}_aD_x^{-n} f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\mathrm{d}t. \tag{3.4.9} $$

It is evident that the integral in (3.4.9) is meaningful for any number n provided its real part is greater than zero.

3.4.2. Riemann-Liouville fractional integrals of order α

Notation 3.4.2. ${}_aI_x^{\alpha},\ {}_aD_x^{-\alpha},\ I_{a+}^{\alpha}$**; Riemann-Liouville left-sided fractional integral of order** α**.**

Notation 3.4.3. ${}_xI_b^{\alpha},\ {}_xD_b^{-\alpha}, I_{b-}^{\alpha}$**; Riemann-Liouville right-sided fractional integral of order** α**.**

Definition 3.4.2. **Let** $f(x) \in L(a,b), \alpha \in C\ (\Re(\alpha) > 0)$, **then**

$$ {}_aI_x^{\alpha} f(x) = {}_aD_x^{-\alpha} f(x) = I_{a+}^{\alpha}\ f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)\mathrm{d}t}{(x-t)^{1-\alpha}},\ \ x > a \tag{3.4.10} $$

is called Riemann-Liouville left-sided fractional integral of order α.

Definition 3.4.3. **Let** $f(x) \in L(a,b), \alpha \in C\ (\Re(\alpha) > 0)$, **then**

$$ {}_xI_b^{\alpha}\ f(x) = {}_xD_b^{-\alpha}\ f(x) = I_{b-}^{\alpha}\ f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)\mathrm{d}t}{(t-x)^{1-\alpha}}, x < b \tag{3.4.11} $$

is called Riemann-Liouville right-sided fractional integral of order α.

Example 3.4.1. If $f(x) = (x-a)^{\beta-1}$, then find the value of ${}_aI_x^{\alpha} f(x)$.

Solution: We have

$$ {}_aI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}(t-a)^{\beta-1}\mathrm{d}t $$

If we substitute $t = a + y(x-a)$ in the above integral, it reduces to

$$ \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1} $$

where $\Re(\beta) > 0$. Thus

$$ {}_aI_x^{\alpha} f(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}. \tag{3.4.12} $$

Example 3.4.2. It can be similarly shown that

$$ {}_xI_b^\alpha g(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(b-x)^{\alpha+\beta-1}, \quad x < b, \tag{3.4.13}$$

where $\Re(\beta) > 0$ and $g(x) = (b-x)^{\beta-1}$.

Note 3.4.1. It may be noted that (3.4.12) and (3.4.13) give the Riemann-Liouville integrals of the power functions $f(x) = (x-a)^{\beta-1}$ and $g(x) = (b-x)^{\beta-1}$, $\Re(\beta) > 0$.

3.4.3. Basic properties of fractional integrals

Property 3.4.1. *Fractional integrals obey the following property:*

$$\begin{aligned} {}_aI_x^\alpha\ {}_aI_x^\beta\ \phi &= {}_aI_x^{\alpha+\beta}\ \phi = {}_aI_x^\beta\ {}_aI_x^\alpha\phi, \\ {}_xI_b^\alpha\ {}_xI_b^\beta\ \phi &= {}_xI_b^{\alpha+\beta}\ \phi = {}_xI_b^\beta\ {}_xI_b^\alpha\ \phi. \end{aligned} \tag{3.4.14}$$

Proof. By virtue of the definition (3.4.10), it follows that

$$\begin{aligned} {}_aI_x^\alpha\,{}_aI_x^\beta\phi &= \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\mathrm{d}t}{(x-t)^{1-\alpha}}\frac{1}{\Gamma(\beta)}\int_a^t \frac{\phi(u)\mathrm{d}u}{(t-u)^{1-\beta}} \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x \mathrm{d}u\ \phi(u)\int_u^x \frac{\mathrm{d}t}{(x-t)^{1-\alpha}(t-u)^{1-\beta}} \end{aligned} \tag{3.4.15}$$

If we use the substitution $y = \frac{t-u}{x-u}$, the value of the second integral is

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)(x-u)^{1-\alpha-\beta}}\int_0^1 y^{\beta-1}(1-y)^{\alpha-1}\mathrm{d}y = \frac{(x-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)},$$

which, when substituted in (3.4.15) yields the first part of (3.4.14). The second part can be similarly established. In particular,

$$ {}_aI_x^{n+\alpha} f = {}_aI_x^n\,{}_aI_x^\alpha f, \quad (n \in \mathbb{N}, \Re(\alpha) > 0) \tag{3.4.16}$$

which shows that the n-fold differentiation

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\ {}_aI_x^{n+\alpha} f(x) = {}_aI_x^\alpha f(x), \quad (n \in \mathbb{N}, \Re(\alpha) > 0) \tag{3.4.17}$$

for all x. When $\alpha = 0$, we obtain

$$ {}_aI_x^0 f(x) = f(x);\ {}_aI_x^{-n} f(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) = f^{(n)}(x). \tag{3.4.18}$$

Note 3.4.2. The property given in (3.4.14) is called semigroup property of fractional integration.

Notation 3.4.4. $L(a,b)$, **space of Lebesgue measurable real or complex valued functions.**

Definition 3.4.4. $L(a,b)$**, consists of Lebesgue measurable real or complex valued functions** $f(x)$ **on** $[a,b]$**:**

$$L(a,b) = \{f : \|f\|_1 \equiv \int_a^b |f(t)|\mathrm{d}t < +\infty\}. \tag{3.4.19}$$

Note 3.4.3. The operators ${}_aI_x^\alpha$ and ${}_xI_b^\alpha$ are defined on the space $L(a,b)$.

Property 3.4.2. *The following results holds.*

$$\int_a^b f(x)({}_aI_x^\alpha g)\mathrm{d}x = \int_a^b g(x)({}_xI_b^\alpha f)\mathrm{d}x. \tag{3.4.20}$$

(3.4.20) can be established by interchanging the order of integration in the integral on the left-hand side of (3.4.20) and then using the Dirichlet formula (3.4.4).

The above property is called the property of "integration by parts" for fractional integrals.

3.4.4. A useful integral

We now evaluate the following integral given by Saxena and Nishimoto [*Journal of Fractional calculus*, Vol.6, (1994), 65 -75].

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ct+d)^\gamma \mathrm{d}t = (ac+d)^\gamma (b-a)^{\alpha+\beta-1} \times B(\alpha,\beta)\,{}_2F_1\left[\alpha, -\gamma; \alpha+\beta; \frac{(a-b)c}{ac+d}\right], \tag{3.4.21}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, |\arg \frac{d+bc}{d+ca}| < \pi, a, c$ and d are constants.

Solution: Let

$$\begin{aligned} I &= \int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ct+d)^\gamma \mathrm{d}t \\ &= (ac+d)^\gamma \sum_{k=0}^\infty \frac{(-1)^k(-\gamma)_k c^k}{(ac+d)^k} \int_a^b (t-a)^{\alpha+k-1}(b-t)^{\beta-1}\mathrm{d}t \\ &= (ac+d)^\gamma (b-a)^{\alpha+\beta-1} B(\alpha,\beta)\ {}_2F_1\left(-\gamma, -\alpha; \alpha+\beta; \frac{(a-b)c}{ac+d}\right). \end{aligned}$$

In evaluating the inner integral, the modified form of the beta function, namely

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta), \tag{3.4.22}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0$, is used.

Example 3.4.3. As a consequence of (3.4.21), it follows that

$$\begin{aligned} {}_aI_x^\alpha[(x-a)^{\beta-1}(cx+d)^\gamma] &= (ac+d)^\gamma(x-a)^{\alpha+\beta-1}\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\ &\times {}_2F_1(\beta,-\gamma;\alpha+\beta;\frac{(a-x)c}{ac+d}), \end{aligned} \tag{3.4.23}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, |\arg \frac{(a-x)c}{ac+d}| < \pi; a, c$ and d being constants. In a similar manner we obtain the following result.

Example 3.4.4. We also have

$$\begin{aligned} {}_xI_b^\alpha[(b-x)^{\beta-1}(cx+d)^\gamma] &= (cx+d)^\gamma(b-x)^{\alpha+\beta-1}\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\ &\times {}_2F_1(\alpha,-\gamma;\alpha+\beta;\frac{(x-b)c}{cx+d}), \end{aligned} \tag{3.4.24}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, |\arg \frac{(x-b)c}{cx+d}| < \pi$.

Example 3.4.5. On the otherhand if we set $\gamma = -\alpha-\beta$ in (3.4.21) it is found that

$$\begin{aligned} &{}_aD_x^{-\alpha}[(x-a)^{\beta-1}(cx+d)^{-\alpha-\beta}] \\ &= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(ac+d)^{-\alpha}(x-a)^{\alpha+\beta-1}(d+cx)^{-\beta}, \end{aligned} \tag{3.4.25}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0$.

Example 3.4.6. Similarly, we have

$$\begin{aligned} &{}_xI_b^\alpha[(b-x)^{\beta-1}(cx+d)^{-\alpha-\beta}] \\ &= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(cx+d)^{-\beta}(bc+d)^{-\alpha}(b-x)^{\alpha+\beta-1} \end{aligned} \tag{3.4.26}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0$.

3.4.5. The Weyl integral

Notation 3.4.5. ${}_xW^{\alpha}_{\infty}, {}_xI^{\alpha}_{\infty}$, **Weyl integral of order** α.

Definition 3.4.5. **The Weyl fractional integral of $f(x)$ of order α, denoted by ${}_xW^{\alpha}_{\infty}$, is defined by**

$$ {}_xW^{\alpha}_{\infty} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t)\mathrm{d}t, (-\infty < x < \infty) \tag{3.4.27} $$

where $\alpha \in C, \Re(\alpha) > 0$. (3.4.27) is also denoted by $I^{\alpha}_{-} f(x)$.

Example 3.4.7. Prove that

$$ {}_xW^{\alpha}_{\infty} \mathrm{e}^{-\lambda x} = \frac{\mathrm{e}^{-\lambda x}}{\lambda^{\alpha}}, \text{ where } \Re(\alpha) > 0. \tag{3.4.28} $$

Solution: We have

$$ \begin{aligned} {}_xW^{\alpha}_{\infty} \mathrm{e}^{-\lambda x} &= \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} \mathrm{e}^{-\lambda t}\mathrm{d}t, \lambda > 0 \\ &= \frac{\mathrm{e}^{-\lambda x}}{\Gamma(\alpha)\lambda^{\alpha}} \int_0^{\infty} u^{\alpha-1}\mathrm{e}^{-u}\mathrm{d}u \\ &= \frac{\mathrm{e}^{-\lambda x}}{\lambda^{\alpha}}, \Re(\alpha) > 0. \end{aligned} $$

Notation 3.4.6. ${}_xD^{\alpha}_{\infty}, D^{\alpha}_{-}$ **:** **Weyl fractional derivative.**

Definition 3.4.6. : **The Weyl fractional derivative of order α, denoted by ${}_xD^{\alpha}_{\infty}$ is defined by**

$$ \begin{aligned} {}_xD^{\alpha}_{\infty} f(x) = D^{\alpha}_{-} f(x) &= (-1)^m \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \left({}_xW^{m-\alpha}_{\infty} f(x)\right) \\ &= (-1)^m \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \frac{1}{\Gamma(m-\alpha)} \int_x^{\infty} \frac{f(t)\mathrm{d}t}{(t-x)^{1+\alpha-m}}, \quad (-\infty < x < \infty) \end{aligned} \tag{3.4.29} $$

where $m - 1 \leq \alpha < m, m \in N, \alpha \in c$.

Example 3.4.8. Find ${}_xD^{\alpha}_{\infty}\mathrm{e}^{-\lambda x}, \lambda > 0$.

Solution: We have

$$
\begin{aligned}
{}_xD^{\alpha}_{\infty}\, e^{-\lambda x} &= (-1)^m \left(\frac{d}{dx}\right)^m {}_xW^{m-\alpha}_{\infty} e^{-\lambda x} \\
&= (-1)^m \left(\frac{d}{dx}\right)^m \lambda^{-(m-\alpha)} e^{-\lambda x} \\
&= \lambda^{\alpha}\, e^{-\lambda x}.
\end{aligned}
\tag{3.4.30}
$$

3.4.6. Basic properties of Weyl integral

Property 3.4.3 : *The following relation holds.*

$$
\int_0^{\infty} \phi(x)\Big({}_0I^{\alpha}_x \psi(x)\Big)dx = \int_0^{\infty} \Big({}_xW^{\alpha}_{\infty}\phi(x)\Big)\psi(x)dx. \tag{3.4.31}
$$

(3.4.31) is called the formula for fractional integration by parts. It is also caled Parseval equality. (3.4.31) can be established by interchanging the order of integration.

Property 3.4.4 : *Weyl fractional integral obeys the semigroup property. That is,*

$$
\Big({}_xW^{\alpha}_{\infty}\ {}_xW^{\beta}_{\infty} f\Big) = \Big({}_xW^{\alpha+\beta}_{\infty} f\Big) = \Big({}_xW^{\beta}_{\infty}\ {}_xW^{\alpha}_{\infty} f\Big). \tag{3.4.32}
$$

Proof. We have

$$
\begin{aligned}
{}_xW^{\alpha}_{\infty}\ {}_xW^{\beta}_{\infty} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^{\infty} dt(t-x)^{\alpha-1} \\
&\times \frac{1}{\Gamma(\beta)} \int_t^{\infty} (u-t)^{\alpha-1} f(u)du.
\end{aligned}
$$

Using the modified form of Dirichlet formula (3.4.4), namely

$$
\begin{aligned}
&\int_x^{a} dt(t-x)^{\alpha-1} \int_t^{a} (u-t)^{\beta-1} f(u)du \\
&= B(\alpha,\beta) \int_t^{a} (u-t)^{\alpha+\beta-1} f(u)du,
\end{aligned}
\tag{3.4.33}
$$

and letting $a \to \infty$, (3.4.33) yields the desired result

$$
\Big({}_xW^{\alpha}_{\infty}\ {}_xW^{\beta}_{\infty} f\Big) = \Big({}_xW^{\alpha+\beta}_{\infty} f\Big). \tag{3.4.34}
$$

Notation 3.4.7. ${}_{-\infty}W_x^\alpha, I_+^\alpha$**: Weyl integral with lower limit** $-\infty$**.**

Definition 3.4.7. Another companion to the operator (3.4.27) is the following:

$$ {}_{-\infty}W_x^\alpha f(x) = I_+^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad (-\infty < x < \infty) \tag{3.4.35}$$

where $\alpha \in c, \quad (\Re(\alpha) > 0)$.

Note 3.4.4 : The operator defined by (3.4.34) is useful in fractional diffusion problems of physics and related areas.

Example 3.4.9. Prove that

$$ {}_{-\infty}W_x^\alpha \mathrm{e}^{ax} = \frac{\mathrm{e}^{ax}}{a^\alpha}. \tag{3.4.36}$$

Solution: We have, by setting $x - t = u$.

Note 3.4.5 : An alternative form of (3.4.34) in terms of convolution is given by

$$ {}_{-\infty}W_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} t_+^{\alpha-1} f(x-t)\mathrm{d}t \tag{3.4.37}$$

where

$$ t_+^{\alpha-1} = \begin{cases} t^{\alpha-1}, t > 0 \\ 0, t < 0 \end{cases} \tag{3.4.38}$$

Example 3.4.10. Prove that

$$ {}_xW_\infty^\upsilon(\cos ax) = a^{-\upsilon}\cos(ax + \frac{1}{2}\pi\upsilon) \tag{3.4.39}$$

where $a > 0, 0 < \Re(\upsilon) < 1$.

Solution: The result follows from the known integral

$$ \int_u^\infty (x-u)^{\upsilon-1}\cos ax\,\mathrm{d}x = \frac{\Gamma(\upsilon)}{a^\upsilon}\cos(au + \frac{\upsilon\pi}{2}) \tag{3.4.40}$$

where $a > 0, 0 < \Re(\upsilon) < 1$.

Example 3.4.11. Prove that

$$ {}_xW_\infty^{\upsilon}(\sin ax) = a^{-\upsilon}\sin(ax + \frac{1}{2}\pi\upsilon) \tag{3.4.41}$$

Hint : Use the integral

$$\int_u^\infty (x-u)^{\upsilon-1}\sin ax \,\mathrm{d}x = \frac{\Gamma(\upsilon)}{a^{\upsilon}}\sin(au + \frac{\upsilon\pi}{2}) \tag{3.4.42}$$

where $a > 0, 0 < \Re(\upsilon) < 1$.

Exercises 3.4.

3.4.1. Prove that

$$\left({}_aI_x^{\alpha}(x-a)^{\beta-1}\right) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}, \;\; \Re(\beta) > 0.$$

3.4.2. Prove that

$$\left({}_aI_x^{\alpha}(x\pm c)^{\gamma-1}\right) = \frac{(a\pm c)^{\gamma-1}}{\Gamma(\alpha+1)}(x-a)^{\alpha}\;{}_2F_1\left(1, 1-\gamma;\; \alpha+1;\; \frac{a-x}{a\pm c}\right)$$

where $\Re(\beta) > 0, \gamma \in \mathbb{C}, a < x < b$.

3.4.3. Prove that

$$\left({}_aI_x^{\alpha}\left[(x-a)^{\beta-1}(b-x)^{\gamma-1}\right]\right) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{(x-a)^{\alpha+\beta-1}}{(b-a)^{1-\gamma}} \times {}_2F_1\left(\beta, 1-\gamma; \alpha+\beta; \frac{x-a}{b-a}\right)$$

where $\Re(\beta) > 0, \gamma \in \mathbb{C}, a < x < b$.

3.4.4. Prove that

$$\left({}_aI_x^{\alpha}\left[\frac{(x-a)^{\beta-1}}{(b-x)^{\alpha+\beta}}\right]\right) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{(x-a)^{\alpha+\beta-1}}{(b-a)^{\alpha}(b-x)^{\beta}}$$

$\Re(\beta) > 0, a < x < b$.

3.4.5. Prove that

$$\left({}_aI_x^{\alpha}\left[(x-a)^{\beta-1}(x\pm c)^{\gamma-1}\right]\right) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{(x-a)^{\alpha+\beta-1}}{(a\pm c)^{1-r}} \times {}_2F_1\left(\beta, 1-\gamma; \alpha+\beta; \frac{a-x}{a\pm c}\right),$$

where $\Re(\beta) > 0, \gamma \in \mathbb{C}, a \pm c > 0$.

3.4.6. Prove that

$$\left({}_aI_x^{\alpha}\left[\frac{(x-a)^{\beta-1}}{(x\pm c)^{\alpha+\beta}}\right]\right) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{(x-a)^{\alpha+\beta-1}}{(a\pm c)^{\alpha}(x\pm c)^{\beta}}$$

3.4.7. Prove that

$$\left({}_aI_x^{\alpha}[\mathrm{e}^{\lambda x}]\right) = \mathrm{e}^{\lambda a}(x-a)^{\alpha}\ E_{1,\alpha+1}(\lambda x - \lambda a).$$

where $x > a$.

3.4.8. Prove that

$$\left({}_aI_x^{\alpha}[\mathrm{e}^{\lambda x}(x-a)^{\beta-1}]\right) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\mathrm{e}^{\lambda a}(x-a)^{\alpha+\beta-1}\ {}_1F_1\left(\beta; \alpha+\beta; \lambda x - \lambda a\right),$$

where $\Re(\beta) > 0$.

3.4.9. Prove that

$$\left({}_aI_x^{\alpha}\left[(x-a)^{\beta-1}\ \ln(x-a)\right]\right) = (x-a)^{\alpha+\beta-1}\ \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}[\ln(x-a)+\psi(\beta)-\psi(\alpha+\beta)].$$

where $\Re(\beta) > 0$, where $\psi(.)$ is the logarithmic derivative of the gamma function.

3.4.10. Prove that

$$\left({}_aI_x^{\nu}\left[(x-a)^{\frac{\nu}{2}}\ \ J_{\nu}(\lambda\sqrt{x-a})\right]\right) = \left(\frac{2}{\lambda}\right)^{\nu}(x-a)^{\frac{a+\nu}{2}}\ \ J_{a+\nu}(\lambda\sqrt{x-a}),$$

where $\Re(\nu) > -1$.

3.4.11. Prove that

$$\left({}_aI_x^\nu\ \left[(x-a)^{\beta-1}\ {}_2F_1(\mu,\nu;\beta;\lambda(x-a))\right]\right) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\nu+\beta-1}\ {}_2F_1\ (\mu,\nu;\nu+\beta;\lambda x-\lambda a),$$

where $\Re(\beta) > 0$.

3.4.12. Prove that

$$\left({}_aI_x^\nu\ \left[(x-a)^{\beta-1}\ E_{\mu,\beta}\Big((x-a)^\mu\Big)\right]\right) = (x-a)^{\nu+\beta-1}\ E_{\mu,\nu+\beta}\Big[(x-a)^\mu\Big].$$

3.4.13. Show that

$$\left({}_aI_x^\nu\Big[x^{\mu-1}\sin ax\Big]\right) = \frac{x^{\mu+\nu-1}\Gamma(\mu)}{2i\quad\Gamma(\mu+\nu)} \times\Big[{}_1F_1(\mu;\mu+\nu;\text{iax}) - {}_1F_1(\mu;\mu+\nu;-\text{iax})\Big],$$

where $a > 0, \Re(\nu) > 0, \Re(\mu) > 0$.

3.4.14. Establish the formula

$$\left({}_aI_x^\nu\Big[x^{\mu-1}\cos ax\Big]\right) = \frac{x^{\mu+\nu-1}\Gamma(\mu)}{2\Gamma(\mu+\nu)} \times\Big[{}_1F_1(\mu;\mu+\nu;\text{iax}) - {}_1F_1(\mu;\mu+\nu;-\text{iax})\Big],$$

where $a > 0, \Re(\nu) > 0, \Re(\mu) > 0$.

3.4.15. Prove the following results:

$$\left({}_aI_x^\nu \begin{Bmatrix}\sin\lambda(x-a)\\ \cos\lambda(x-a)\end{Bmatrix}\right) = \frac{i^{-(1\pm1)/2}}{2\Gamma(\nu+1)}(x-a)^\nu \times\Big[{}_1F_1(1;\nu+1;i\lambda(x-a)) \mp {}_1F_1(1;\nu+1;-i\lambda(x-a))\Big]$$

3.4.16. Prove the following results:

$$\left({}_aI_x^\nu\left[(x-a)^{-\frac{1}{2}}\begin{Bmatrix}\cos\ \lambda\sqrt{x-a}\\ \sin\ \lambda\sqrt{x-a}\end{Bmatrix}\right]\right)$$
$$=\ \sqrt{\pi}\left(\frac{\lambda}{2}\right)^{\frac{1}{2}-\nu}(x-a)^{(2\nu-1)/4}\begin{Bmatrix}J_{\nu-\frac{1}{2}}(\lambda\sqrt{x-a})\\ I_{\nu-\frac{1}{2}}(\lambda\sqrt{x-a})\end{Bmatrix}.$$

3.4.17. Prove that

$$\left({}_0I_x^\nu\left[x^{\alpha-1}{}_2F_1(a,b;c;-wx)\right]\right)$$
$$=\frac{B(\alpha,\nu)}{\Gamma(\nu)}x^{\alpha+\nu-1}{}_3F_2(a,b,\alpha;c,\alpha+\nu;-wx),$$

where $x, \Re(\alpha), \Re(\nu) > 0, |\arg\ (1+wy)| < \pi$.

3.4.18. Prove that

$$\left({}_0I_x^\nu\left[x^{\alpha-1}{}_2F_1(a,b;c;\ 1-wx)\right]\right)=\frac{x^{(\alpha,\nu-1)}}{\Gamma(\nu)}\Gamma\begin{bmatrix}c,\ c-a-b,\ \alpha,\nu\\ c-a,\ c-b,\ \alpha+\nu\end{bmatrix}$$
$$\times\,{}_3F_2(a,b,\alpha;\ a+b-c+1,\ \alpha+\nu;\ wx)$$
$$+\ \frac{w^{c-a-b}}{\Gamma(\nu)}x^{c-a-b+\alpha+\nu-1}\ \Gamma\begin{bmatrix}c,\ a+b-c,\ c-a-b+\alpha\\ a,\ b,\ c-a-b+\alpha+\nu\end{bmatrix}$$
$$\times\,{}_3F_2(c-a,\ c-b,\ c-a-b+\alpha;\ c-a-b+1;\ c-a-b+\alpha+\nu;\ wx)$$

where $x, \Re(\alpha), \Re(\nu), \Re(c-a-b+\alpha) > 0, |\arg\ w| < \pi$ and $\Gamma\begin{bmatrix}a\ b\ c\\ d\ e\ f\end{bmatrix}$ stands for the ratio of product of gamma functions $\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}{\Gamma(d)\,\Gamma(e)\,\Gamma(f)}$.

3.4.19. Prove that

$$\left({}_0I_x^\nu\left[x^{c-1}(1-xz)^{-\rho}{}_2F_1(a,b;c;wx)\right]\right)=\frac{\Gamma(c)}{\Gamma(\nu+c)}x^{c+\nu-1}(1-xz)^{-\rho}$$
$$\times F_3\left(\rho,a,\nu,b,c+\nu;\frac{xz}{xz-1},\ wx\right),$$

where $x, \Re(c), \Re(\nu) > 0$; $|\arg\ (1-wx)| < \pi$ and $|\arg\ (1-z)| < \pi$; F_3 is the Appell's hypergeometric function of two variables, defined by

$$F_3(a,a';b,b';c;y,z)=\sum_{k,l=0}^{\infty}\frac{(a)_k\ (a')_l\ (b)_k\ (b')_l y^k z^l}{(c)_{k+l}k!l!}.$$

3.4.20. Prove that

$$ {}_aI_x^\nu\left[\frac{(x-a)^{\alpha-1}}{x-y}\right] = \frac{(x-a)^{\alpha+\nu-1}\Gamma(\alpha)}{(x-y)\Gamma(\alpha+\nu)}\,{}_2F_1\left(1,\nu;\alpha+\nu;\frac{x-a}{x-y}\right) $$

where $\Re(\alpha), \Re(\beta) > 0;\ y < a < x$.

3.4.21. Prove that

$$ \left({}_0I_x^\nu\left[x^{\alpha-1}(1-ux)^{-\rho}(1-vx)^{-\lambda}\right]\right) = \frac{x^{\alpha+\nu-1}\Gamma(\alpha)}{\Gamma(\alpha+\nu)}F_1(\alpha,\rho,\lambda,\alpha+\nu;ux,vx), $$
$$ x|u| < 1,\quad |\arg\ u| < \pi; |\arg\ v| < \pi, x|v| < 1; $$

where $x, \Re(\alpha), \Re(\nu) > 0$. F_1 is the Appell's hypergeometric function of the two variables defined by

$$ F_1(a,b,b';c;y,z) = \sum_{k,l=0}^{\infty}\frac{(a)_{k+l}(b)_l(b')_l y^k z^l}{(c)_{k+l}k!l!}. $$

3.4.22. Prove that

$$ \left({}_0I_x^\nu\left[x^{\alpha-1}\ln^n(cx+d)\right]\right) = \frac{x^{\alpha+\nu-1}\Gamma(\alpha)}{\Gamma(\alpha+\nu)}\frac{\partial^n}{\partial\rho^n}\left[d^\rho\,{}_2F_1\left(-\rho,\alpha;\alpha+\nu;-\frac{xc}{\mathrm{d}}\right)\right]_{\rho=0}, $$

where $x, \Re(\alpha), \Re(\beta) > 0; |\arg\ (cx+d)| < \pi$ and $0 \le t \le x$.

3.4.23. Prove that

$$ \left({}_0I_x^\nu\left[\ln(cx+d)\right]\right) = \frac{1}{\Gamma(\nu)}\int_0^x t^{\nu-1}\ln(cx+d-ct)\mathrm{d}x $$

where $\Re(\nu) > 0$.

3.4.24. Prove that

$$ \left({}_aI_x^\nu\left[x^{\alpha-1}\ln(cx+d)\right]\right) = \frac{x^{\alpha+\nu-1}\Gamma(\alpha)}{\Gamma(\alpha+\nu)}\ln \mathrm{d} $$
$$ + c\mathrm{d}^{-1}x^{\alpha+\nu}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\nu+1)}\,{}_3F_2(\alpha+1,1,1;2,\alpha+\nu+1;-\frac{cx}{d}), $$

where $x, \Re(\alpha), \Re(\nu) > 0; |\arg\ (cx+d)| < \pi$ and $0 \le t \le x$.

3.4.25. Prove that

$$\left({}_0I_x^{\nu}\left[\mathrm{e}^{\lambda x}\right]\right) = x^{\nu}E_{1,\nu+1}(\lambda x), (\Re(\alpha) > 0).$$

3.4.26. Prove that

$$\left({}_0I_x^{\nu}\left[\frac{\sinh\sqrt{\lambda}x}{\sqrt{\lambda}x}\right]\right) = x^{\nu+1}E_{2,\nu+2}(\lambda x^2),\ \ (\Re(\alpha) > 0).$$

3.4.27. Prove that

$$\left({}_0I_x^{\nu}\left[\lambda, E_{\nu}(\lambda x^{\nu})\right]\right) = E_{\nu}(\lambda x^{\nu}) - 1,\ \ \Re(\nu) > 0.$$

3.4.28. Prove that

$$\left({}_0I_x^{\nu}\left[x^{\rho-1}(x+a)^{\sigma}\right]\right) = a^{\sigma}x^{\rho+\nu-1}\frac{\Gamma(\rho)}{\Gamma(\rho+\nu)}{}_2F_1\left(-\sigma,\rho;\nu+\rho;-\frac{x}{a}\right),$$

where $\Re(\nu) > 0; \Re(\rho) > 0, |\arg\,(\frac{x}{a})| < \pi$.

3.4.29. Prove that

$$\left({}_0I_x^{\nu}\left[x^{\rho-1}(x^k+a^k)^{\sigma}\right]\right) = a^{k\sigma}x^{\nu+\rho-1}\frac{\Gamma(\rho)}{\Gamma(\rho+\nu)}$$
$$\times\,{}_{k+1}F_k\left(-\sigma,\Delta(k;\lambda);\Delta(k;\rho+\nu);-\frac{x^k}{a^k}\right),$$

where $k \in \mathbb{N}, \Re(\nu) > 0; \Re(\rho) > 0,\ \ |\arg\,(\frac{x}{a})| < \frac{\pi}{k}$, and $\Delta(k;a)$ represents the sequence of parameters $\frac{a}{k}, \frac{a+1}{k} \cdots, \frac{a+k-1}{k}$.

3.4.30. Prove that

$$\left({}_0I_x^{\nu}\left[x^{\rho-1}\exp(ax^k)\right]\right) = \frac{\Gamma(\rho)}{\Gamma(\rho+\nu)}x^{\rho+\nu-1}{}_kF_k\left(\Delta(k;\rho);\Delta(k;\rho+\nu);ax^k\right),$$

where $\Re(\nu) > 0; \Re(\rho) > 0, k = 2, 3, 4, \cdots$.

3.4.31.

$$\left({}_0I_x^{\nu}\left[x^{\rho-1}{}_pF_q(a_1,\cdots,a_p;\rho,b_2,\cdots,b_q;ax)\right]\right)=\frac{x^{\rho+\nu-1}\Gamma(\rho)}{\Gamma(\nu+\rho)}$$
$$\times\,{}_pF_q(a_1,\cdots,a_p;\rho+\nu,b_2,\cdots,b_q;ax)$$

where $p\le q+1, \Re(\nu)>0; \Re(\rho)>0, |ax|<1$ if $p=q+1$.

3.4.32.

$$\left({}_0I_x^{\nu}\left[x^{\rho-1}G_{p,q}^{m,n}\left(ax\Big|_{b_1,\cdots,b_q}^{a_1,\cdots,a_p}\right)\right]\right)=x^{\rho+\nu-1}G_{p+1,q+1}^{m,n+1}\left(ax\Big|_{b_1,\cdots,b_q;1-\rho-\nu}^{1-\rho,a_1,\cdots,a_p}\right)$$

where $G_{p,q}^{m,n}(.)$ is Meiger's G-Function;

$$m+n>\frac{1}{2}(p+q), |\arg a|<\left(m+n-\frac{p}{2}-\frac{q}{2}\right)\pi$$
$$\Re(\rho+b_j)>0, j=1,\cdots,m, \Re(\nu)>0.$$

3.4.33. Prove that

$$\left({}_0I_x^{\alpha}\left[x^{\rho-1}H_{p,q}^{m,n}\left(ax\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right)\right]\right)=x^{\rho+\alpha-1}H_{p+1,q+1}^{m,n+1}\left[ax\Big|_{(b_q,B_q),\,(1-\rho-\alpha,1)}^{(1-\rho,1),\,(a_p,A_p)}\right]$$

where $\Re\left[\rho+\min_{1\le j\le m}\left(\frac{b_j}{B_j}\right)\right]>0, \Re(\alpha)>0, |\arg a|<\frac{1}{2}\pi c^*$;

$$c^*=\sum_{j=1}^{n}A_j-\sum_{j=n+1}^{p}A_j+\sum_{j=1}^{m}B_j-\sum_{j=m+1}^{q}B_j>0$$

and $\mu^*=\sum_{j=1}^{p}A_j-\sum_{j=1}^{q}B_j<0$ or $\mu^*=0$ and $0<|ax|<\beta^{-1}$;

$$\beta=\left\{\prod_{j=1}^{p}(A_j)^{A_j}\right\}\left\{\prod_{j=1}^{q}(B_j)^{-B_j}\right\}.$$

3.4.34. Prove that

$$\left({}_0I_x^{\alpha}\left[x^{\rho-1}H_{p,q}^{m,n}\left(ax^{\lambda}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right)\right]\right)=x^{\rho+\alpha-1}H_{p+1,q+1}^{m,n+1}\left[ax^{\lambda}\Big|_{(b_q,B_q),(1-\rho-\alpha,\lambda)}^{(1-\rho,\lambda),(a_p,A_p)}\right]$$

where $\Re\left[\rho+\lambda\min_{1\le j\le m}\left(\frac{b_j}{B_j}\right)\right]>0, \Re(\alpha)>0, \lambda>0|\arg a|<\frac{1}{2}\pi c^*; c^*$ is defined in Exercise 3.4.33 above,

$$c^*>0; \mu^*<0 \text{ or } \mu^*=0 \text{ and } 0<|ax^{\lambda}|<\beta^{-1};$$

$$\beta = \prod_{j=1}^{p}(A_j)^{A_j}\prod_{j=1}^{q}(B_j)^{-B_j};\ \mu^* \text{ is given in Exercise 3.4.33.}$$

3.4.35. Prove that

$$\left({}_xI_\infty^\nu\left[x^{\alpha-1}(x-u)^{-\rho}(x-v)^{-\lambda}\right]\right) = \frac{x^{\alpha+\nu-\lambda-\rho-1}\Gamma(1+\lambda+\rho-\alpha-\nu)}{\Gamma(1+\lambda+\rho-\alpha)} \times F_1(1+\lambda+\rho-\alpha-\nu,\rho,\lambda,1+\lambda+\rho-\alpha;u/x,v/x),$$

where $x > 0$; $|\arg u| < \pi$ and $|u| < x$; $|\arg v| < \pi$ and $|v| < x$; $0 < \Re(\nu) < \Re(\lambda+\rho-\alpha)+1$.

3.4.36. Prove that

$$\left({}_xI_\infty^\nu\left[x^{\alpha-1}\ln(cx+d)\right]\right) = x^{\alpha+\nu-1}\frac{\Gamma(1-\alpha-\nu)}{\Gamma(1-\alpha)}\left[\ln(cx)+\psi(1-\alpha)-\psi(1-\alpha-\nu)\right] + \mathrm{d}c^{-1}x^{\alpha+\nu-2}\frac{\Gamma(2-\alpha-\nu)}{\Gamma(2-\alpha)}{}_3F_2(2-\alpha-\nu,1,1;2,2-\alpha;-\frac{\mathrm{d}}{cx})$$

where $x, \Re(\beta) > 0, \Re(\alpha+\nu) < 1; |\arg\ (cx+\mathrm{d})| < \pi$ and $x \le t < \infty$.

3.4.37. Prove that

$$\left({}_xW_\infty^\nu\left[x^{-\lambda}(x+a)^\mu\right]\right) = \frac{x^{\mu+\nu-\lambda}\Gamma(\lambda-\mu-\nu)}{\Gamma(\lambda-\mu)} \times {}_2F_1(-\mu,\lambda-\mu-\nu;\lambda-\mu;-a/x)$$

where $0 < \Re(\nu) < \Re(\lambda-\mu)$; $|\arg\ (a/x)| < \pi$ or $|(a/x)| < 1, \Re(\nu) > 0$.

3.4.38. Prove that

$$\left({}_xW_\infty^\nu\left[x^{\nu-1}\mathrm{e}^{-ax}\right]\right) = \pi^{-\frac{1}{2}}(x/a)^{\nu-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}ax}K_{\nu-\frac{1}{2}}\left(\frac{1}{2}ax\right)$$

where $\Re(\nu) > 0, \Re(ax) > 0$.

3.4.39. Prove that

$$\left({}_xW_\infty^\nu\left[x^{-\lambda}\mathrm{e}^{-ax}\right]\right) = a^{B-\frac{1}{2}}x^{-B-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}ax}\ W_{A,B}(ax)$$

where $2A = 1-\lambda-\nu, 2B = \lambda-\nu; \Re(ax) > 0, \Re(\nu) > 0$.

3.4.40. Prove that

$$\left({}_xW_\infty^\nu\left[x^{-2\nu}e^{\frac{a}{x}}\right]\right) = \left(\frac{\pi}{x}\right)^{\frac{1}{2}} a^{\frac{1}{2}-\nu}\exp\left(\frac{a}{2x}\right)I_{\nu-\frac{1}{2}}\left(\frac{a}{2x}\right),$$

where $\Re(\nu) > 0$.

3.4.41. Prove that

$$\left({}_xW_\infty^\nu\left[x^{-\lambda}\exp\left(\frac{a}{x}\right)\right]\right) = \frac{\Gamma(\lambda-\nu)}{\Gamma(\lambda)}x^{\nu-\lambda}{}_1F_1\ (\lambda-\nu;\lambda;\frac{a}{x}),$$

where $0 < \Re(\nu) < \Re(\lambda)$.

3.4.42. Prove that

$$\left({}_xW_\infty^\nu\left[\exp\left(-ax^{\frac{1}{2}}\right)\right]\right) = 2^{\nu+\frac{1}{2}}\pi^{-\frac{1}{2}}a^{\frac{1}{2}-\nu}x^{\frac{1}{2}\nu+\frac{1}{4}}K_{\nu+\frac{1}{2}}\left(ax^{\frac{1}{2}}\right),$$

where $\Re(ax^{\frac{1}{2}}) > 0, \Re(\nu) > 0$.

3.4.43. Prove that

$$\left({}_xW_\infty^{+\nu}\left[x^{-\frac{1}{2}}\exp\left(-ax^{\frac{1}{2}}\right)\right]\right) = 2^{\nu+\frac{1}{2}}\pi^{-\frac{1}{2}}x^{\frac{1}{2}-\nu}x^{\frac{1}{2}\nu-\frac{1}{4}}K_{\nu-\frac{1}{2}}\left(ax^{\frac{1}{2}}\right),$$

where $\Re(ax^{\frac{1}{2}}) > 0, \Re(\nu) > 0$.

3.4.44. Prove that

$$\left({}_xW_\infty^\nu\left[x^{-\lambda}\log x\right]\right) = \frac{\Gamma(\lambda-\nu)}{\Gamma(\lambda)}\ x^{\nu-\lambda}\ [\log x + \psi(\lambda) - \psi(\lambda-\nu)],$$

where $0 < \Re(\nu) < \Re(\lambda)$.

3.4.45. Prove that

$$\left({}_xW_\infty^\nu\left[x^{-\frac{\mu}{2}}J_\mu\left(ax^{\frac{1}{2}}\right)\right]\right) = 2^\nu a^{-\nu}x^{\frac{1}{2}\nu-\frac{1}{2}\mu}J_{\mu-\nu}\left(ax^{\frac{1}{2}}\right),$$

$a, x > 0;\ 0 < \Re(\nu) < \frac{1}{2}\Re(\mu) + \frac{3}{4}$.

3.4.46. Prove that

$$\left({}_xW_\infty^\nu\left[x^\lambda J_\mu\left(ax^{\frac{1}{2}}\right)\right]\right) = 2^{2\lambda}a^{-2\lambda}x^\nu\ G_{1,3}^{2,0}\left(\frac{a^2x}{4}\Big|_{-\nu,\lambda+\frac{1}{2}\mu,\lambda-\frac{1}{2}\mu}^{0}\right),$$

where $a > 0, x > 0,\ 0 < \Re(\nu) < \frac{1}{4} - \Re(\lambda)$.

3.4.47. Prove that

$$\left({}_xW_\infty^\alpha\left[x^{\lambda-1}{}_2F_1(a,b;c;1-wx)\right]\right)=\frac{w^{-a}}{\Gamma(\alpha)}x^{\lambda+\alpha-a-1}\,\Gamma\left[\begin{matrix}c,\ b-a,\ \alpha,\ a-\lambda-\alpha+1\\ b,\ c-a,\ a-\lambda+1\end{matrix}\right]$$
$$\times\,{}_3F_2\,(a,\ c-b,\ a-\lambda-\alpha+1;\ a-\lambda+1,\ a-b+1;\ \frac{1}{wx})$$
$$+\frac{w^{-b}}{\Gamma(\alpha)}x^{\lambda+\alpha-b-1}\Gamma\left[\begin{matrix}c,\ a-b,\ \alpha,\ b-a-\alpha+1\\ a,\ c-b,\ b-\lambda+1\end{matrix}\right]$$
$$\times\,{}_3F_2\,(b,\ c-a,\ b-a-\alpha+1;\ b-a+1,\ b-\lambda+1;\ \frac{1}{wx})$$

where $x,\ \Re(\alpha)>0, \Re(\lambda+\alpha-a)>1,\ \Re(\lambda+\alpha-b)<1;\ |\arg w|<\pi.$

3.4.48. Prove that

$$\left({}_xW_\infty^\nu\left[x^{\lambda-1}{}_2F_1(a,b;c;-wx)\right]\right)=\frac{1}{\Gamma(\nu)}x^{\alpha+\nu-1}B(\nu,1-\lambda-\nu)$$
$$\times\,{}_3F_2\,(a,\ b,\ \lambda;c,\lambda+\nu;\ -wx)+\frac{w^{1-\lambda-\nu}}{\Gamma(\nu)}\ \Gamma\left[\begin{matrix}c,\ a-\lambda-\nu+1,\ b-\lambda-\nu+1,\ \lambda+\nu-1\\ a,\ b,\ c-\lambda-\nu+1\end{matrix}\right]$$
$$\times\,{}_3F_2\,(1-\nu,\ a-\lambda-\nu+1,b-\lambda-\nu+1;\ 2-\lambda-\nu,c-\lambda-\nu+1;-wx),$$

where $x,\ \Re(\nu)>0;\Re(\lambda+\nu-a),\ \Re(\nu-b)<1;\ |\arg w|<\pi.$

3.4.49. Prove that

$$\left({}_xW_\infty^\nu\left[x^{\alpha-1}G_{p,q}^{m,n}\left(ax\Big|_{b_1,\cdots,b_q}^{a_1,\cdots,a_p}\right)\right]\right)=x^{\alpha+\nu-1}G_{p+1,q+1}^{m+1,n}\left[ax\Big|_{1-\alpha-\nu,b_1,\cdots,b_q}^{a_1,\cdots,a_p,1-\alpha}\right]$$

where $\Re(\nu)>0, \Re[\alpha+\nu+\max_{1\le j\le n}(a_j)]<2\quad |\arg a|<\frac{1}{2}\pi c^*,\ c^*>0.$ c^* is defined in Exercise 3.4.33

3.4.50. Prove that

$$\left({}_xI_\infty^\nu\left[x^{\alpha-1}H_{p,q}^{m,n}\left(ax^\lambda\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right)\right]\right)=x^{\alpha+\nu-1}H_{p+1,q+1}^{m+1,n}\left[ax^\lambda\Big|_{(1-\alpha-\nu,\lambda),(b_q,B_q)}^{(a_p,A_p),(1-\alpha,\lambda)}\right]$$

where $\lambda>0,\ \Re(\nu)>0,\ |\arg a|<\frac{1}{2}\pi c^*,\ c^*>0,\ c^*$ is defined in Exercise 3.4.33; $\Re[\alpha+\nu+\max_{1\le j\le n}\frac{(a_j-1)}{A_j}]<1.$

3.5. Derivatives of Fractional Order

In this section, we study various fractional order derivatives which occur in certain reaction (relaxation) and diffusion problems.

3.5.1. Riemann - Liouville fractional derivatives of arbitrary order

Notation 3.5.1. $\{\alpha\}$ **means the fractional part of number** $\alpha,\ 0 \leq \{\alpha\} < 1$**.**

Notation 3.5.2. $[\alpha]$ **means the integral part of number** α.

Note 3.5.1 We note that

$$\alpha = \{\alpha\} + [\alpha]. \tag{3.5.1}$$

Notation 3.5.3. $D^{\alpha}_{a+},\ {}_aD^{\alpha}_x\phi(x)$**, Riemann - Liouville derivative of the function** $\phi(x)$ **of order** α, **(left - hand).**

Notation 3.5.4. $D^{\alpha}_{b-},\ {}_bD^{\alpha}_x\phi(x)$**, Riemann - Liouville derivative of the function** $\phi(x)$ **of order** α, **(right-hand).**

Definition 3.5.1. The left-hand Riemann - Liouville derivative of order $\alpha > 0$ **is defined by**

$$D^{\alpha}_{a+}\phi(x) = {}_aD^{\alpha}_x\phi(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_a^x \frac{\phi(t)\mathrm{d}t}{(x-t)^{\alpha-n+1}}\ (n = [\alpha]+1). \tag{3.5.2}$$

Example 3.5.1. Prove that

$${}_0D^{\alpha}_x x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}x^{\gamma-\alpha},\ \alpha \geq 0,\ \gamma > -1,\ x > 0. \tag{3.5.3}$$

Solution: We have

$$ {}_0D_x^\alpha(x^\gamma) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_0^x t^\gamma(x-t)^{n-\alpha-1}\mathrm{d}t $$
$$ = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+n-\alpha)}(\gamma-\alpha+1)_n x^{\gamma-\alpha} $$

for $\gamma + 1 > 0, \gamma + 1 + n - \alpha > 0$

$$ = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}\ x^{\gamma-\alpha}, $$

for $\gamma + 1 > 0,\ (\gamma - \alpha + 1)_n \neq 0.$

Note 3.5.2 It is interesting to observe that for $\gamma = 0$, We obtain

$$ {}_0D_x^\alpha\,(1) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha \neq 1, 2, \cdots . \tag{3.5.4} $$

which is a remarkable result and indicates that fractional derivative of a constant is not zero.

Definition 3.5.2. The right - hand Riemann - Liouville fractional derivative of order α, of the function $\phi(x)$ is defined by

$$ D_{b-}^\alpha\phi(x) = {}_xD_b^\alpha\phi(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_x^b \frac{\phi(t)\mathrm{d}t}{(t-x)^{\alpha-n+1}}\ (n = [\alpha]+1). \tag{3.5.5} $$

In short, we can express (3.5.2) in the form

$$ {}_aD_x^\alpha\phi(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\ {}_aI_x^{n-\alpha}\phi(x) \tag{3.5.6} $$

and (3.5.3) as

$$ {}_xD_b^\alpha\phi(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\ {}_xI_b^{n-\alpha}\phi(x). \tag{3.5.7} $$

We shall also employ the notations

$$ {}_aD_x^\alpha\phi = {}_aI_x^{-\alpha}\phi = \left({}_aI_x^\alpha\right)^{-1}\phi, \alpha \geq 0. \tag{3.5.8} $$

Similarly, we have

$$ {}_xD_b^\alpha\phi = {}_bI_x^{-\alpha}\phi = \left({}_bI_x^\alpha\right)^{-1}\phi, \alpha \geq 0. \tag{3.5.9} $$

Example 3.5.2. Prove that

$$ {}_0I_x^\nu \ \ln x = \frac{x^\nu}{\Gamma(\nu+1)}[\ln x - \gamma - \psi(\nu+1)]. $$

Solution: We have

$$ {}_0I_x^\nu \ \ln x = \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1} \ln t \, \mathrm{d}t $$

If we make the change of variable $t = xu$, then

$$ \begin{aligned} {}_0I_x^\nu \ \ln x &= \frac{1}{\Gamma(\nu)}\int_0^1 x^\nu(1-u)^{\nu-1}(\ln x + \ln u)\mathrm{d}u \\ &= \frac{x^\nu \ln(x)}{\Gamma(\nu+1)} + \frac{x^\nu}{\Gamma(\nu)}\int_0^1 (1-u)^{\nu-1} \ln u \, \mathrm{d}u. \end{aligned} $$

We know that

$$ \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \ln t \, \mathrm{d}t = B(\alpha,\beta)[\psi(\alpha) - \psi(\alpha+\beta)], $$

where $\Re(\alpha) > 0, \Re(\beta) > 0$. Applying the above formula for $\alpha = 1$, and noting that $\psi(1) = -\gamma$, we see that

$$ {}_0I_x^\nu \ \ln x = \frac{x^\nu}{\Gamma(\nu+1)}[\ln x - \gamma - \psi(\nu+1)]. $$

Similarly we can establish the result in the next exercise.

Example 3.5.3. Prove that

$$ {}_0D_x^\alpha \ (\ln x) = \frac{x^{-\nu}}{\Gamma(1-\nu)}[\ln x - \gamma - \psi(-\nu+1)] $$

Example 3.5.4. Prove that

$$ {}_0D_x^\alpha \ (\mathrm{e}^{ax}) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}{}_1F_1(1; 1-\alpha; ax) $$

Solution: We have

$$
\begin{aligned}
{}_0D_x^{\alpha}\,(\mathrm{e}^{ax}) &= \sum_{r=0}^{\infty} \frac{a^r}{r!}\,{}_0D_x^{\alpha}(x^r) \\
&= \sum_{r=0}^{\infty} \frac{a^r}{r!} \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} x^{r-\alpha} \\
&= \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\,{}_1F_1(1; 1-\alpha; x).
\end{aligned}
$$

Example 3.5.5. Prove that

$$
{}_0D_x^{\alpha}\,(x+a)^p = \frac{a^p x^{-\alpha}}{\Gamma(1-\alpha)}\,{}_2F_1(1, -p; 1-\alpha; -\frac{x}{a}).
$$

Solution: We have

$$
\begin{aligned}
{}_0D_x^{\alpha}\,(x+a)^p &= a^p \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{(-p)_r\,{}_0D_x^{\alpha}(x^r)}{a^r} \\
&= a^p \sum_{r=0}^{\infty} \frac{(-1)^r (-p)_r \Gamma(r+1)}{r! a^r \Gamma(r-\alpha+1)} x^{r-\alpha} \\
&= \frac{a^p x^{-\alpha}}{\Gamma(1-\alpha)}\,{}_2F_1(1, -p; 1-\alpha; -\frac{x}{a}).
\end{aligned}
$$

In above expression, the following result has been used:

$$
{}_0D_x^{\alpha}\,(x^p) = \frac{\Gamma(p+1) x^{p-\alpha}}{\Gamma(p-\alpha+1)}.
$$

Exercises 3.5.

3.5.1. Prove that

$$
\left({}_0D_x^{\alpha}\,[\sin ax]\right) = \frac{x^{-\alpha}}{2i\Gamma(1-\alpha)} \Big[{}_1F_1(1; 1-\alpha; iax) - {}_1F_1(1; 1-\alpha; -iax)\Big].
$$

3.5.2. Prove that

$$\left({}_0D_x^\alpha\ [\cos ax]\right) = \frac{x^{-\alpha}}{2\Gamma(1-\alpha)}\Big[{}_1F_1(1;1-\alpha;iax) + {}_1F_1(1;1-\alpha;-iax)\Big].$$

3.5.3. Prove that

$$\left({}_0D_x^\alpha\ [x^p \ln x]\right) = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} x^{p-\alpha}[\ln x + \psi(p+1) - \psi(p-\alpha+1)]$$

where $\psi(.)$ is the digamma function.

3.5.4. Prove that

$$\left({}_0D_x^\alpha\ [x^p(a+x)^q]\right) = \frac{\Gamma(p+1)x^{p-\alpha}a^q}{\Gamma(p-\alpha+1)}\, {}_2F_1(-q, p+1; p-\alpha+1; -\frac{x}{a}),$$

for $(p-\alpha+1)_n \neq 0, p \neq -1, -2, -3, \cdots, |\frac{x}{a}| < 1$.

3.5.5. Prove that

$$\left({}_0D_x^\alpha\ [x^p e^{ax}]\right) = \frac{\Gamma(p+1)x^{p-\alpha}}{\Gamma(p-\alpha+1)}\, {}_1F_1(p+1; p-\alpha+1; ax)$$

for $(p-\alpha+1+k)_n \neq 0, k = 0, 1, \cdots, p \neq -1, -2, -3, \cdots,$

Special functions can be expressed as fractional derivatives. This can be seen from the following exercises.

3.5.6. Prove that

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)} x^{1-c}\, {}_0D_x^{b-c}[x^{b-1}(1-x)^{-a}]$$

for $\Re(c-b) > 0, \Re(b) > 0$.

3.5.7. Prove that

$${}_1F_1(a; c; x) = \phi(a, c; x) = \frac{\Gamma(c)}{\Gamma(a)} x^{1-c}\, {}_0D_x^{a-c}(e^x x^{a-1})$$

for $\Re(a-c) > 0, \Re(c) > 0$.

3.5.8. Prove that

$$ {}_{p+1}F_{q+1}\left[\begin{matrix} a_1,\cdots,a_p \\ b_1,\cdots,b_q \end{matrix}; x\right] = \frac{\Gamma(d)}{\Gamma(c)} x^{1-d} {}_0D_x^{c-d} \left\{ x^{c-1} {}_pF_q\left[\begin{matrix} a_1,\cdots,a_p \\ b_1,\cdots,b_q \end{matrix}; x\right]\right\} $$

for $\Re(c-d) > 0, \Re(d) > 0$.

3.5.9. Prove that

$$ J_\nu(x) = \pi^{-\frac{1}{2}} 2^{1-\nu} x^{-\nu} {}_0D_x^{-\nu+\frac{1}{2}} (\sin x). $$

3.5.10. Prove that

$$ \psi(x) = -\gamma + \ln z - \Gamma(x) z^{1-x} {}_0D_z^{1-x} (\ln z). $$

3.5.11. Prove that

$$ \gamma(a, z) = \Gamma(a) \mathrm{e}^{-z} {}_0D_z^{-a} (\mathrm{e}^z) $$

where $\gamma(a, z)$ is incomplete gamma function.

3.5.12. Prove that

$$ \left({}_0D_x^\nu \left[x^{\frac{\mu}{2}} J_\mu(x^{\frac{1}{2}}) \right] \right) = 2^{-\nu} x^{\frac{1}{2}(\mu-\nu)} J_{\mu-\nu}(x^{\frac{1}{2}}), $$

where $\Re(\mu) > -1$.

3.5.13. Prove that

$$ \left({}_0D_x^\nu \left[\sin x^{\frac{1}{2}} \right] \right) = \frac{1}{2} \pi^{\frac{1}{2}} (2x^{\frac{1}{2}})^{\frac{1}{2}-\nu} J_{\frac{1}{2}-\nu}(x^{\frac{1}{2}}). $$

3.5.14. Prove that

$$ \left({}_0D_x^\nu \left[x^{-\frac{1}{2}} \cos x^{\frac{1}{2}} \right] \right) = \sqrt{\pi}\, (2x^{\frac{1}{2}})^{-\nu-\frac{1}{2}} J_{-\nu-\frac{1}{2}}(x^{\frac{1}{2}}). $$

3.5.15. Prove that

$$ \left({}_0D_x^\nu \left[x^{\frac{\mu}{2}} I_\mu(x^{\frac{1}{2}}) \right] \right) = 2^{-\nu} x^{\frac{1}{2}(\mu-\nu)} I_{\mu-\nu}(x^{\frac{1}{2}}). $$

3.5.16. Show that

$$\left({}_0D_x^\nu\left[\sinh x^{\frac{1}{2}}\right]\right) = \frac{1}{2}\sqrt{\pi}\,(2x^{\frac{1}{2}})^{\frac{1}{2}-\nu} I_{\frac{1}{2}-\nu}(x^{\frac{1}{2}})$$

and

$$\left({}_0D_x^\nu\left[\cosh x^{\frac{1}{2}}\right]\right) = \sqrt{\pi}\,(2x^{\frac{1}{2}})^{-\frac{1}{2}-\nu} I_{-\frac{1}{2}-\nu}(x^{\frac{1}{2}}).$$

3.5.17. Prove that

$$\left({}_0D_x^\nu\left[x^\lambda {}_1F_0(\alpha; -; x)\right]\right) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\nu+1)} x^{\lambda-\nu} \times {}_2F_1(\lambda+1, \alpha, \lambda-\nu+1; x).$$

3.5.18. Establish the result

$${}_0D_x^\nu\left[x^\lambda {}_2F_1(a, b; c; x)\right] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\nu+1)} x^{\lambda-\nu} \times {}_3F_2(\lambda+1, a, b, c; \lambda-\nu+1; x)$$

where $\Re(\lambda) > -1, \Re(\lambda+1-\nu) > 0$, and $c \neq 0, -1, -2, \cdots$ and $|x| < 1$.

3.5.19. Prove that

$${}_0D_x^\nu\left[x^\lambda {}_pF_q(a_1, \cdots a_p; b_1, \cdots b_q; x)\right] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\nu+1)} x^{\lambda-\nu} \times {}_{p+1}F_{q+1}(\lambda+1, a_1, \cdots a_p; b_1, \cdots b_q, \lambda-\nu+1; x)$$

where $\Re(\lambda) > -1, \Re(\lambda-\nu+1) > 0$.

3.5.20. Prove that

$$\left({}_0D_x^\nu\left[x^\lambda {}_pF_q(a_1, \cdots a_p; b_1, \cdots b_q; ax^2)\right]\right) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\nu+1)} x^{\lambda-\nu} \times {}_{p+2}F_{q+2}\left[\frac{1}{2}(\lambda+1), \frac{1}{2}(\lambda+2), a_1, \cdots, a_p; b_1, \cdots, b_q, \frac{1}{2}(\lambda-\nu+1), \frac{1}{2}(\lambda-\nu+2); ax^2\right]$$

where $\Re(\lambda) > -1, \Re(\lambda-\nu+1) > 0$ and $|ax^2| < 1; b_j \neq 0, -1, -2, \cdots$ $(j = 1, \cdots, p)$.

3.5.21. Show that

$$\left({}_0D_x^\nu\ [e^{ax^2}]\right) = x^{-\nu}\sum_{n=0}^{\infty}\frac{\Gamma(2n+1)(ax^2)^n}{\Gamma(2n-\nu+1)n!}.$$

3.5.22. Show that

$$\begin{aligned}\left({}_0D_x^\nu\ J_\mu[x]\right) = &\frac{x^{\mu-\nu}}{2\mu\Gamma(\mu-\nu+1)}\\ &\times {}_2F_3\Big[\frac{1}{2}(\mu+1),\frac{1}{2}(\mu+2);\mu+1,\frac{1}{2}(\mu-\nu+1),\\ &\frac{1}{2}(\mu-\nu+2);-\Big(\frac{1}{2}x\Big)^2\Big].\end{aligned}$$

3.5.23. Prove that

$$\begin{aligned}{}_{p+1}F_q\left[{}^{-n,\alpha_1,\cdots,\alpha_p;x}_{\beta_1,\cdots,\beta_q}\right] = &\frac{\Gamma(\beta_1)\cdots\Gamma(\beta_q)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}x^{1-\beta_q}\\ &\times {}_0D_x^{\alpha_p-\beta_q}\left(x^{\alpha_p-\beta_{q-1}}\right){}_0D_x^{\alpha_{p-1}-\beta_{q-1}}\left(x^{\alpha_{p-1}-\beta_{q-2}}\right)\cdots\\ &\times x^{\alpha_3-\beta_2}{}_0D_x^{\alpha_2-\beta_2}x^{\alpha_2-\beta_1}{}_0D_x^{\alpha_1-\beta_1}\left[x^{\alpha_1-1}(1-x)^n\right].\end{aligned}$$

3.5.24. Prove that ${}_0I_x^\nu(e^\lambda x) = x^\nu E_{1,\nu+1}(\lambda x)$ where $\Re(\alpha) > 0$.

3.5.25. Prove that $\left({}_0I_x^\nu[\cosh\ \sqrt{\lambda}\ x]\right) = x^\nu E_{2,\nu+1}(\lambda x^2)$, where $\Re(\alpha) > 0$.

3.5.26. Prove that $\left({}_0I_x^\nu\left[\frac{\sinh\ \sqrt{\lambda}\ x}{\sqrt{\lambda}\ x}\right]\right) = x^{\nu+1}E_{2,\nu+2}(\lambda x^2)$.

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