CHAPTER 4

APPLICATIONS IN ASTROPHYSICS AND STOCHASTIC PROCESSES

[The physics part of this chapter is based on the lectures of Professor Dr. Hans Haubold of the Outer Space Division of the United Nations and the stochastic process part is based on the lectures of Dr. R.N. Pillai, Dr. K.K. Jose, Dr. V. Seethalekshmi, Dr. Alice Thomas, Dr. K. Jayakumar, Dr. E. Sandhya and Dr. S. Satheesh.]

4.0. Introduction

Since the participants of the Third S.E.R.C. school are of different backgrounds it is not worth discussing the individual topics in astrophysics and stochastic processes. Only the mathematical aspects where the various special functions come in naturally will be discussed here. We start with a few topics in astrophysics such as solar models, energy generations in stars, gravitational instability problems and reaction-diffusion problems. As stochastic process problems, we will only deal with Mittag-Leffler process and the associated applications.

4.1. Solar Models

When looking at the internal structure of the Sun one has to look at mass conservation, hydrostatic equilibrium, energy conservation and energy transport. These can be described by a system of non-linear differential equations which cannot be fully solved analytically. Hence the standard techniques adopted are numerical evaluations thereby resulting in computer-generated solar models. In order to derive analytic models a starting point would be to look into the density distribution in the core of the Sun. It is well known that the matter density decreases from the center to the exterior, temperature and pressure also behave the same way. The solar core is a more stabilized region and hence an analytic model for the matter density distribution in the solar core is an appropriate starting point. Let r be an arbitrary distance from the center of the Sun and $R_\odot$ the solar radius. The simplest model for matter density $\rho(r)$ is a linear model of the type

$$\rho(r) \ = \ \rho_0 \, [1 - \frac{r}{R_\odot}] \qquad\qquad (\text{ Model } 1 \text{ })$$

which indicates that the matter density decreases linearly from the core to the surface and it is zero at the surface and it is a constant ρ_0 at the center. But observations indicate that a linear model is not correct. A more appropriate starting point would be to consider the non-linear model

$$\rho(r) \ = \ \rho_0 \left[1 - \left(\frac{r}{R_\odot}\right)^\delta\right], \ \ \delta > 0 \qquad (\text{ Model } 2 \text{ }) \tag{4.1.1}$$

where δ is an arbitrary parameter. Since the surface area of a sphere of radius r is $4\pi r^2$ the mass of the Sun can be computed from the relation.

$$\frac{\mathrm{d}}{\mathrm{d}r} M(r) = 4\pi r^2 \rho(r). \tag{4.1.2}$$

That is,

$$\begin{aligned} M(r) &= 4\pi\rho_0 \int_0^r t^2 \left[1 - \left(\frac{t}{R_\odot}\right)^\delta\right] \mathrm{d}t \\ &= \frac{4\pi}{3}\rho_0 r^3 \left[1 - \frac{3}{(\delta+3)}\left(\frac{r}{R_\odot}\right)^\delta\right]. \end{aligned} \tag{4.1.3}$$

Denoting the total mass by $M_\odot$, this gives the central density

$$\rho_0 = \frac{3}{4\pi} \frac{(\delta+3)}{\delta} \frac{M_\odot}{R_\odot}. \tag{4.1.4}$$

From the connection between pressure $P(r)$, mass $M(r)$ and density $\rho(r)$, namely,

$$\frac{\mathrm{d}}{\mathrm{d}r} P(r) = G \frac{M(r)\rho(r)}{r^2} \tag{4.1.5}$$

where G is the gravitational constant, we have

$$
\begin{aligned}
P(r) &= P(0) - G\int_0^r \frac{M(t)\rho(t)}{t^2}\mathrm{d}t \\
&= \frac{4\pi G}{3}\rho_0^2 R_\odot^2 \left\{\xi - \frac{1}{2}\left(\frac{r}{R_\odot}\right)^2 + \frac{(\delta+6)}{(\delta+2)(\delta+3)}\left(\frac{r}{R_\odot}\right)^{\delta+2}\right. \\
&\quad \left. - \frac{3}{2(\delta+1)(\delta+3)}\left(\frac{r}{R_\odot}\right)^{2\delta+2}\right\}
\end{aligned}
\tag{4.1.6}
$$

where

$$
\xi = \frac{1}{2} - \frac{(\delta+6)}{(\delta+2)(\delta+3)} + \frac{3}{2(\delta+1)(\delta+3)}. \tag{4.1.7}
$$

From the relationship between temperature $T(r)$ and pressure $P(r)$, namely,

$$
T(r) = \frac{\mu}{kN_A}\frac{P(r)}{\rho(r)} \tag{4.1.8}
$$

where μ is the mean molecular weight, N_A is Avogadro's constant and k is the Boltzmann constant, we have

$$
\begin{aligned}
T(r) = {}& \frac{4\pi G\,\mu\,\rho_0}{3kN_A}\frac{R_\odot^2}{[1-(\frac{r}{R_\odot})^\delta]}\left\{\xi - \frac{1}{2}\left(\frac{r}{R_\odot}\right)^2\right. \\
&\left. + \frac{(\delta+6)}{(\delta+2)(\delta+3)}\left(\frac{r}{R_\odot}\right)^{\delta+2} - \frac{3}{2(\delta+1)(\delta+3)}\left(\frac{r}{R_\odot}\right)^{2\delta+2}\right\}.
\end{aligned}
\tag{4.1.9}
$$

From the equation of energy conservation, namely,

$$
\frac{\mathrm{d}}{\mathrm{d}r}L(r) = 4\pi r^2\rho(r)\epsilon(r) \tag{4.1.10}
$$

where $L(r)$ represents the energy flux through the sphere with radius r so that $L(R_\odot)$ represents the luminosity of the Sun and $\epsilon(r)$ is the rate of thermonuclear energy generation per unit mass including the tiny energy losses via solar neutrinos, we can compute luminosity once an expression is available for $\epsilon(r)$.

If we consider a specific reaction, say particles 1 and 2 reacting to give rise to particles 3 and 4 or $1 + 2 \to 3 + 4$ then the internal luminosity due to this specific reaction is given by

$$L_{12}(R_\odot) = \int_0^{R_\odot} 4\pi r^2 \rho(r)\epsilon_{12}(r)\,\mathrm{d}r \tag{4.1.11}$$

A general model that can be used to represent $\epsilon(r)$ is the following:

$$\epsilon(r) = \epsilon_0 \left[\frac{\rho(r)}{\rho_0}\right]^\alpha \left[\frac{T(r)}{T_0}\right]^\beta, \tag{4.1.12}$$

where α and β are real constants.

4.1.1. A more general model for density

A more general two-parameter model for the matter density distribution is the following:

$$\rho(r) = \rho_0 \left[1 - \left(\frac{r}{R_\odot}\right)^\delta\right]^\gamma, \ \delta > 0, \gamma > 0 \qquad \text{(Model 3)} \tag{4.1.13}$$

Example 4.1.1. Evaluate the total mass $M(R_\odot)$ as well as the mass contained in a sphere of arbitrary radius r under the model in (4.1.13).

Solution: The total mass is given by

$$M_\odot = \int_0^{R_\odot} 4\pi r^2 \rho_0 \left[1 - \left(\frac{r}{R_\odot}\right)^\delta\right]^\gamma \mathrm{d}r \tag{4.1.14}$$

$$= 4\pi R_\odot^3 \rho_0 \int_0^1 x^2 \left[1 - x^\delta\right]^\gamma \mathrm{d}x, \ x = \frac{r}{R_\odot}$$

$$= 4\pi R_\odot^3 \rho_0 \int_0^1 y^{\frac{3}{\delta}-1} [1-y]^\gamma \mathrm{d}y, \ y = x^\delta$$

$$= 4\pi R_\odot^3 \rho_0 \frac{\Gamma\left(\frac{3}{\delta}\right)\Gamma(\gamma+1)}{\Gamma(\gamma+1+\frac{3}{\delta})}, \tag{4.1.15}$$

evaluating (4.1.14) with the help of a type-1 beta integral. Mass at an arbitrary radius r is given by

$$
\begin{aligned}
M(r) &= \int_0^r 4\pi t^2 \rho_0 \left[1 - \left(\frac{t}{R_\odot}\right)^{\delta}\right]^{\gamma} \mathrm{d}t \\
&= 4\pi\rho_0 R_\odot^3 \int_0^{r/R_\odot} u^2 \left[1 - u^{\delta}\right]^{\gamma} \mathrm{d}u,\ u = \frac{t}{R_\odot}, 0 \le u \le 1.
\end{aligned}
$$

Expanding $(1 - u^\delta)^\gamma$ with the help of a binomial expansion and then integrating out we have the following:

$$
\begin{aligned}
(1 - u^\delta)^\gamma &= (1 - u^\delta)^{-(-\gamma)} = \sum_{k=0}^{\infty} \frac{(-\gamma)_k}{k!} (u^\delta)^k. \\
M(r) &= 4\pi\rho_0 R_\odot^3 \sum_{k=0}^{\infty} \frac{(-\gamma)_k}{k!} \int_0^{r/R_\odot} u^{2+\delta k}\, \mathrm{d}u \\
&= 4\pi\rho_0 R_\odot^3 \sum_{k=0}^{\infty} \frac{(-\gamma)_k}{k!} \frac{(r/R_\odot)^{3+\delta k}}{3+\delta k} \\
&= \frac{4\pi\rho_0}{\delta} \sum_{k=0}^{\infty} \frac{(-\gamma)_k}{k!} \frac{[(r/R_\odot)^\delta]^k}{k + \frac{3}{\delta}}.
\end{aligned}
$$

But

$$
\frac{1}{k + \frac{3}{\delta}} = \frac{1}{\frac{3}{\delta}} \frac{(\frac{3}{\delta})_k}{(\frac{3}{\delta} + 1)_k}.
$$

Hence

$$
M(r) = \frac{4\pi\rho_0}{3}\ {}_2F_1(-\gamma, \frac{3}{\delta}; \frac{3}{\delta} + 1; (r/R_\odot)^\delta). \tag{4.1.16}
$$

Note that the models in (4.1.13) and (4.1.1) are really valid only in the interior core of the Sun. The convective zone has entirely a different behavior for the density distribution. But for computing the total mass, pressure, temperature and luminosity we have integrated out over the entire length of the solar radius $R_\odot$. This is not appropriate. The integration should have been done only in the interior core of the Sun. Hence a more appropriate model is of the following form:

$$\rho(r) = \rho_0\ [1 - a(\frac{r}{R_\odot})^\delta]^\gamma, \quad \delta > 0, \gamma > 0, a > 0 \qquad \text{(Model 4).} \tag{4.1.17}$$

Since $1 - ax^\delta > 0$, $x = \frac{r}{R_\odot}$, we have $0 \le x \le \frac{1}{a^{\frac{1}{\delta}}}$. Hence for the total integral the range should have been $0 \le r \le \frac{R_\odot}{a^{\frac{1}{\delta}}}$

Exercises 4.1.

4.1.1. Evaluate ρ_0 in terms of the total mass $M_\odot$ for the Models 1,2,3 and 4.

4.1.2. Evaluate $M(r)$ for the Models 4 in (4.1.17).

4.1.3. Evaluate the expression for pressure $P(r)$ under Models 3 for $\delta = 2$.

4.1.4. Evaluate the expression for temperature under Models 3 for $\delta = 2$.

4.1.5. Evaluate the expression for $M(r)$ under Models 4.

4.2. Solar Thermonuclear Energy Generation

In reaction rate theory when two particles 1 and 2 reacting to give rise to particles 3 and 4, namely $1+2 \to 3+4$, the basic assumption is that the distribution of the relative velocities of the reacting particles always remains Maxwell-Boltzmannian. Then the distribution of the relative velocities of the particles can be written as

$$f(v)\mathrm{d}v = \left(\frac{\mu}{2\pi kT}\right)^{\frac{3}{2}} \exp\left\{-\frac{\mu v^2}{2kT}\right\} 4\pi v^2\ \mathrm{d}v \tag{4.2.1}$$

such that $\int_0^\infty f(v)\mathrm{d}v = 1$. In terms of energy $E = \frac{\mu v^2}{2}$ we have the density for E given by,

$$f(E)\mathrm{d}E = \frac{2}{\pi^{\frac{1}{2}}(kT)^{\frac{3}{2}}} \exp\left\{-\frac{E}{kT}\right\} E^{\frac{1}{2}}\ \mathrm{d}E. \tag{4.2.2}$$

If the relative velocity of the interacting particles 1 and 2 is v then the thermally averaged product of the cross section σ, denoted by the expected value of σv or $< \sigma v >$

has the following expression under Maxwell-Boltzmann velocity distribution (see Mathai and Haubold (1998)) and in the charged particle case.

$$< \sigma v >= \left(\frac{8}{\pi\mu}\right)^{\frac{1}{2}} \sum_{\nu=0}^{2} \frac{S^{(\nu)}(0)}{\nu!} \frac{1}{(kT)^{-\nu+\frac{1}{2}}} \int_0^\infty y^\nu \mathrm{e}^{-y-zy^{-\frac{1}{2}}} \mathrm{d}y. \tag{4.2.3}$$

The reaction probability integral coming from $< \sigma v >$, denoted by $N_\nu(z)$, is then given by

$$N_\nu(z) = \int_0^\infty y^\nu \mathrm{e}^{-y-zy^{-\frac{1}{2}}} \mathrm{d}y. \tag{4.2.4}$$

A more general integral, where (4.2.4) is a particular case, is already evaluated in Example 1.6.3 of Chapter 1. From there we note that

$$N_\nu(z) = \frac{1}{\pi^{\frac{1}{2}}} G_{0,3}^{3,0}[\frac{z^2}{4}|_{0,\frac{1}{2},1+\nu}], \tag{4.2.5}$$

where $G(\cdot)$ is a Meijer's G-function. Equation (4.2.3) is the situation under non-resonant reactions. But with depleted Maxwell-Boltzmann distribution the reaction probability integral changes to the following:

$$N_\nu(z;\delta) = \int_0^\infty y^\nu \mathrm{e}^{-y-zy^{-\frac{1}{2}}} \mathrm{e}^{-y^\delta} \mathrm{d}y. \tag{4.2.6}$$

With modified Maxwell-Boltzmann distribution the reaction probability integral has the following form:

$$N_\nu(z, \mathrm{d}) = \int_0^{\mathrm{d}} y^\nu \mathrm{e}^{-y-zy^{-\frac{1}{2}}} \mathrm{d}y,\ \mathrm{d} < \infty. \tag{4.2.7}$$

In this case it is a fractional integral. In the resonant situation it can be shown (see Mathai and Haubold (1988)) that the reaction probability integral has the following form:

$$N(a, b, q, g) = \int_0^\infty \frac{\mathrm{e}^{-ay-qy^{-\frac{1}{2}}}}{(b-y)^2 + g^2} \mathrm{d}y. \tag{4.2.8}$$

In the corresponding depleted case the above integral will be modified to the following:

$$N(a, b, q, g, \delta) = \int_0^\infty \frac{\mathrm{e}^{-ay-qy^{-\frac{1}{2}}-cy^\delta}}{(b-y)^2 + g^2} \mathrm{d}y. \tag{4.2.9}$$

In the corresponding modified resonant case the integral will have the fractional form

$$N_{\mathrm{d}}(a,b,q,g,\delta)=\int_0^{\mathrm{d}}\frac{\mathrm{e}^{-ay-qy^{-\frac{1}{2}}-cy^{\delta}}}{(b-y)^2+g^2}\mathrm{d}y. \tag{4.2.10}$$

All these various cases and the related situations are considered in a series of papers by Haubold and Mathai, some of the earlier ones are available from Mathai and Haubold (1988). Some of the recent works from 1988 to 2005 will be mentioned in the lectures to be given by Professor Dr. Hans Haubold at the Third S.E.R.C. School.

References

Chandrasekhar, S. (1967). *An Introduction to the Study of Stellar Structure*, Dover Publications, New York.

Mathai, A.M. and Haubold, H.J. (1988). *Modern Problems in Nuclear and Neutrino Astrophysics*, Akademie-Verlag, Berlin.

4.3. Mittag-Leffler Distribution and Processes

4.3.1. Preliminary concepts

In this section we discuss some preliminary concepts which will be of frequent use in this chapter.

Definition 4.3.1. Infinite divisibility.

A random variable x is said to be infinitely divisible if for every $n \in \mathbb{N}$, there exists independently and identically distributed random variables $y_{1n}, y_{2n}, \dots, y_{nn}$ such that $x \stackrel{d}{=} y_{1n} + y_{2n} + \cdots + y_{nn}$, where $\stackrel{d}{=}$ denotes equality in distributions. In terms of distribution functions, a distribution function F is said to be infinitely divisible if for every positive integer n, there exists a distribution function F_n such that $F = \underbrace{F_n \star F_n \star \cdots \star F_n}_{n \text{ times}}$, where $\star$ denotes convolution.

This is equivalent to the existence of a characteristic function $\varphi_n(t)$ for every $n \in \mathbb{N}$ such that $\varphi(t) = [\varphi_n(t)]^n$ where $\varphi(t)$ is the characteristic function of x.

Infinitely divisible distributions occur in various contexts in the modelling of many real phenomena. For instance when modelling the amount of rain x that falls in a period of length T, one can divide x into more general independent parts from the same family. That is,

$$x \stackrel{d}{=} x_{t_1} + x_{t_2-t_1} + \cdots + x_{T-t_{n-1}}.$$

Similarly, the amount of money x paid by an insurance company during a year must be expressible as the sum of the corresponding amounts $x_1, x_2, \ldots, x_{52}$ in each week, That is,

$$x \stackrel{d}{=} x_1 + x_2 + \cdots + x_{52}.$$

A large number of distributions such as normal, exponential, Weibull, gamma, Cauchy, Laplace, logistic, lognormal, Pareto, geometric, Poisson, etc., are infinitely divisible. Various properties and applications of infinitely divisible distributions can be found in Laha and Rohatgi (1979) and Steutel (1979).

4.3.2. Geometric infinite divisibility

The concept of geometric infinite divisibility (g.i.d.) was introduced by Klebanov *et al.* (1984). A random variable y is said to be g.i.d. if for every $p \in (0, 1)$, there exists a sequence of independently and identically distributed random variables $x_1^{(p)}, x_2^{(p)}, \ldots$ such that

$$y \stackrel{d}{=} \sum_{j=1}^{N(p)} x_j^{(p)} \tag{4.3.1}$$

and

$$P\{N(p) = k\} = p(1-p)^{k-1}, \qquad k = 1, 2, \cdots$$

where $y, N(p)$ and $x_j^{(p)}$, $j = 1, 2, \ldots$ are independent. The relation (4.3.1) is equivalent to

$$\begin{aligned}\varphi(t) &= \sum_{j=1}^{\infty} [g(t)]^j \;\; p(1-p)^{j-1} \\ &= \frac{pg(t)}{1-(1-p)g(t)}\end{aligned}$$

where $\varphi(t)$ and $g(t)$ are the characteristic functions of y and $x_j^{(p)}$ respectively.

The class of g.i.d. distributions is a proper subclass of infinitely divisible distributions. The g.i.d. distributions play the same role in 'geometric summation' as infinitely divisible distributions play in the usual summation of independent random variables. Klebanov *et al.* (1984) established that a distribution function F with characteristic function $\varphi(t)$ is g.i.d. if and only if $\exp\left\{1 - \frac{1}{\varphi(t)}\right\}$ is infinitely divisible. Exponential and Laplace distributions are obvious examples of g.i.d. distributions. Pillai (1990b), Mohan *et al.* (1993) discuss properties of g.i.d. distributions. It may be noted that normal distribution is not geometrically infinitely divisible.

4.3.3. Bernstein functions

A C^∞–function f from $(0, \infty)$ to R is said to be completely monotone if $(-1)^n \frac{d^n f}{dx^n} \geq 0$ for all integers $n \geq 0$.

A C^∞–function f from $(0, \infty)$ to R is said to be a Bernstein function, if $f(x) \geq 0$, $x > 0$ and $(-1)^n \frac{d^n f}{dx^n} \leq 0$ for all integers $n \geq 1$. Then f is also referred to as a function with complete monotone derivative (c.m.d).

A completely monotone function is positive, decreasing and convex while a Bernstein function is positive, increasing and concave (see Berg and Forst (1975)).

Fujita (1993) established that a cumulative distribution function G with $G(0) = 0$ is geometrically infinitely divisible, if and only if G can be expressed as

$$G(x) = \sum_{n=1}^{\infty} (-1)^{n+1} W^{n*}([0, x]), \qquad x > 0$$

where $W^{n*}(dx)$ is the n-fold convolution measure of a unique positive measure $W(dx)$ such that

$$\frac{1}{f(x)} = \int_0^\infty e^{-sx} W(ds), \qquad x > 0$$

where $f(x)$ is a Bernstein function, satisfying the conditions

$$\lim_{x \downarrow 0} f(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty.$$

A distribution is said to have complete monotone derivative if its distribution function $F(x)$ is Bernstein. Pillai and Sandhya (1990) proved that the class of distributions having complete monotone derivative is a proper subclass of g.i.d. distributions. This implies that all distributions with complete monotone densities are geometrically infinitely divisible. It is easier to verify the complete monotone criterion and using this approach we can establish the geometric infinite divisibility of many distributions such as Pareto, gamma and Weibull.

The class of non–degenerate generalized gamma convolutions with densities of the form given by

$$f(x) = c\, x^{\beta-1} \prod_{j=1}^{M} (1 + c_j x)^{-r_j}, \qquad x > 0$$

is geometrically infinitely divisible for $0 < \beta \leq 1$. Similarly distributions having densities of the form

$$f(x) = cx^{\beta-1} \exp(-cx^{\alpha}); \qquad 0 < \alpha \leq 1$$

is g.i.d. for $0 < \beta \leq 1$. Also the Bondesson family of distributions with densities of the form

$$f(x) = cx^{\beta-1} \prod_{j=1}^{M} \left[1 + \sum_{k=1}^{N_j} c_{jk} x^{\alpha_{jk}} \right]^{-r_j}$$

is g.i.d. for $0 \leq \beta \leq 1$, $\alpha_{jk} \leq 1$ provided all parameters are strictly positive (see Bondesson(1992)).

4.3.4. Self–decomposability

Let $\{x_n;\ \ n \geq 1\}$ be a sequence of independent random variables, and let $\{b_n\}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \max_{1\leq k\leq n} P\{|x_k| \geq b_n \epsilon\} = 0 \text{ for every } \epsilon > 0.$$

Let $s_n = \sum_{k=1}^{n} x_k$ for $n \geq 1$. Then the class of distributions which are the weak limits of the distributions of the sums $b_n^{-1} s_n - a_n$; $n \geq 1$ where a_n and $b_n > 0$ are suitably chosen constants, is said to constitute class L. Such distributions are called self–decomposable.

A distribution F with characteristic function $\varphi(t)$ is called self–decomposable, if and only if, for every $\alpha \in (0,1)$, there exists a characteristic function $\varphi_\alpha(t)$ such that $\varphi(t) = \varphi(\alpha t)\varphi_\alpha(t) \quad$ for $\ t \in R$.

Clearly, apart from $x \equiv 0$, no lattice random variable can be self–decomposable. All non-degenerate self–decomposable distributions are absolutely continuous.

A discrete analogue of self–decomposability was introduced by Steutel and Van Harn (1979). A distribution on $N_0 \equiv \{0, 1, 2, \ldots\}$ with probability generating function (p.g.f.) $P(z)$ is called discrete self–decomposable if and only if $P(z) = P(1 - \alpha + \alpha z)P_\alpha(z)$; $|z| \leq 1$, $\alpha \in (0,1)$ where $P_\alpha(z)$ is a p.g.f.

If we define $G(z) = P(1 - z)$, then $G(z)$ is called the alternate probability generating function (a.p.g.f.). Then it follows that a distribution is discrete self–decomposable if and only if $G(z) = G(\alpha z)G_\alpha(z)$; $|z| \leq 1$, $\alpha \in (0,1)$ where $G_\alpha(z)$ is some a.p.g.f.

4.3.5. Stable distributions

A distribution function F with characteristic function $\varphi(t)$ is stable if for every pair of positive real numbers b_1 and b_2, there exist finite constants a and $b > 0$ such that $\varphi(b_1 t)\varphi(b_2 t) = \varphi(bt)e^{iat}$ where $i = \sqrt{-1}$.

Clearly, stable distributions are in class L with the additional condition that the random variables x_n; $n \geq 1$ in Subsection 4.3.4 are identically distributed also. F is stable if and only if its characteristic function can be expressed as

$$\ln \varphi(t) = i\alpha t - c|t|^\beta[1 + i\gamma\omega(t,\beta) sgn\ t]$$

where α, β, γ are constants with $c \geq 0$, $0 < \beta \leq 2$, $|\gamma| \leq 1$ and

$$\omega(t,\beta) = \begin{cases} \tan \frac{\pi\beta}{2}; & \beta \neq 1 \\ \frac{2}{\pi} \ln |t|; & \beta = 1. \end{cases}$$

The value $c = 0$ corresponds to the degenerate distribution, and $\beta = 2$ to the normal distribution. The case $\gamma = 0$, $\beta = 1$ corresponds to the Cauchy law (see Laha and Rohatgi (1979)).

4.3.6. Geometrically strictly stable distributions

A random variable y is said to be geometrically strictly stable (g.s.s.) if for any $p \in (0,1)$ there exists a constant $c = c(p) > 0$ and a sequence of independent and identically distributed random variables $y_1, y_2, \ldots$ such that

$$y \stackrel{d}{=} c(p) \sum_{j=1}^{N(p)} y_j$$

where $P\{N(p) = k\} = p(1-p)^{k-1}$; $k = 1, 2, \ldots$ and $y, N(p)$ and y_j; $j = 1, 2, \ldots$ are independent.

If $\varphi(t)$ is the characteristic function of y, then it implies that

$$\varphi(t) = \frac{p\varphi(ct)}{1-(1-p)\varphi(ct)}; \qquad p \in (0,1).$$

Among the geometrically strictly stable distributions, the Laplace distribution and exponential distribution possess all moments. A geometrically strictly stable random variable is clearly geometrically infinitely divisible.

A non–degenerate random variable y is geometrically strictly stable if and only if its characteristic function is of the form

$$\varphi(t) = 1 \Big/ \left[1 + \lambda |t|^{\alpha} \, \exp\left(-i\frac{\pi}{2}\theta\alpha \, sgn\, t\right)\right]$$

where $0 < \alpha \leq 2$, $\lambda > 0$, $|\theta| \leq \min(1, 2/\alpha - 1)$. When $\alpha = 2$, it corresponds to the Laplace distribution. Thus it is apparent that when ordinary summation of random variables is replaced by geometric summation, the Laplace distribution plays the role of the normal distribution, and exponential distribution replaces the degenerate distribution (see Klebanov *et al.*(1984)).

4.3.7. Mittag-Leffler distribution

The Mittag-Leffler distribution was introduced by Pillai (1990a) and has cumulative distribution function given by

$$F_{\alpha}(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{k\alpha}}{\Gamma(1+k\alpha)}; \qquad 0 < \alpha \leq 1; \; x > 0.$$

Its Laplace transform is given by $\phi(t) = \dfrac{1}{1+t^{\alpha}}$; $0 < \alpha \leq 1$; $t \geq 0$; and the distribution may be denoted by ML(α). Here α is called the exponent. It can be regarded as a generalization of the exponential distribution in the sense that $\alpha = 1$, corresponds to the exponential distribution. The Mittag-Leffler distribution is geometrically infinitely divisible and belongs to class L. It is normally attracted to the stable law with exponent α.

If u is exponential with unit mean and y is positive stable with exponent α, then $x = u^{1/\alpha}y$ is distributed as Mittag-Leffler (α). If u is Mittag-Leffler (α) and v is exponential and u and v are independent, then $x = \frac{u}{v}$ is distributed as Pareto type III with survival function $\bar{F}_x(x) = P(x > x) = \frac{1}{1+x^\alpha}$; $0 < \alpha \leq 1$.

For the Mittag-Leffler distribution, $E(x^\delta)$ exists for $0 \leq \delta < \alpha$ and is given by

$$E(x^\delta) = \frac{\Gamma(1-\delta/\alpha)\Gamma(1+\delta/\alpha)}{\Gamma(1-\delta)}.$$

A two parameter Mittag-Leffler distribution can also be defined with the corresponding Laplace transform $\phi(t) = \frac{\lambda^\alpha}{\lambda^\alpha + t^\alpha}$; $0 < \alpha \leq 1$. It may be denoted by $\mathrm{ML}(\alpha, \lambda)$.

Jayakumar and Pillai (1993) considered a more general class called semi–Mittag-Leffler distribution which included the Mittag-Leffler distribution as a special case. A random variable x with positive support is said to have a semi–Mittag-Leffler distribution if its Laplace transform is given by

$$\phi(t) = \frac{1}{1+\eta(t)}$$

where $\eta(t)$ satisfies the functional equation $\eta(t) = a\eta(bt)$ where $0 < b < 1$ and α is the unique solution of $ab^\alpha = 1$. It may be denoted by $\mathrm{SML}(\alpha)$. Then it follows that $\eta(bt) = b^\alpha h(t)$ where $h(t)$ is a periodic function in t with period $\frac{-\ln b}{2\pi\alpha}$. When $h(t)$ is a constant, the distribution reduces to the Mittag-Leffler distribution. The semi–Mittag-Leffler distribution is also geometrically infinitely divisible and belongs to class L.

4.3.8. α–Laplace distribution

The α–Laplace distribution has characteristic function given by $\varphi(t) = \frac{1}{1+|t|^\alpha}$; $0 < \alpha \leq 2$, $-\infty < t < \infty$. This is also called Linnik's distribution. Pillai (1985) refers to it as the α–Laplace distribution since $\alpha = 2$ corresponds to the Laplace distribution. It is unimodal, geometrically strictly stable and belongs to class L. It is normally attracted to the symmetric stable law with exponent α. Also

$$E(|x|^{\delta}) = \frac{2^{\delta} \quad \Gamma\left(1+\frac{\delta}{\alpha}\right)\Gamma\left(1-\frac{\delta}{\alpha}\right)\Gamma((1+\delta)/2)}{\sqrt{\pi}\ \ \Gamma\left(1-\frac{\delta}{2}\right)}$$

where $0 < \delta < \alpha$; $0 < \alpha \leq 2$.

If u and v are independent random variables where u is exponential with unit mean and v is symmetric stable with exponent α, then $x = u^{1/\alpha}v$ is distributed as α–Laplace. Using this result, Devroye (1990) develops an algorithm for generating random variables having α–Laplace distribution.

Pillai (1985) introduced a larger class of distributions called semi–α–Laplace distribution, with characteristic function given by

$$\varphi(t) = \frac{1}{1+\eta(t)}$$

where $\eta(t)$ satisfies the functional equation $\eta(t) = a\eta(bt)$ for $0 < b < 1$ and a is the unique solution of $ab^{\alpha} = 1$, $0 < \alpha \leq 2$. Here b is called the order and α is called the exponent of the distribution. If b_1 and b_2 are the orders of the distribution such that $\dfrac{\ln\, b_1}{\ln\, b_2}$ is irrational, then $\eta(t) = c|t|^{\alpha}$, where c is some constant. Pillai (1985) established that, for a semi–α–Laplace distribution with exponent α, $E|x|^{\delta}$ exists for $0 \leq \delta < \alpha$. It can be shown that

$$\varphi(t) = \frac{1}{1+|t|^{\alpha}[1 - A\cos(k\ \ln\ |t|)]}$$

where $k = \dfrac{2\pi}{\ln\, b}$, $0 < b < 1$ is the characteristic function of a semi–α–Laplace distribution for suitable choice of A and $\alpha < 1$.

The semi–α–Laplace distribution is also geometrically infinitely divisible and belongs to class L. It is useful in modelling household income data. Mohan *et al.*(1993) refer to it as a geometrically right semi–stable law.

4.3.9. Semi–Pareto distribution

The semi–Pareto distribution was introduced by Pillai (1991). A random variable x with positive support has semi–Pareto distribution $SP(\alpha, p)$ if its survival function is given by $\bar{F}_x(x) = P(x > x_0) = \frac{1}{1+\psi(x_0)}$ where $\psi(x_0)$ satisfies the functional equation $p\psi(x) = \psi(p^{1/\alpha}x)$; $0 < p < 1,\ \alpha > 0$.

The above definition is analogous to that of the semi–stable law defined by Levy (see Pillai (1971)). It can be shown that $\psi(x) = x^{\alpha}h(x)$ where $h(x)$ is periodic in $\ln x$ with period $\dfrac{-2\pi\alpha}{\ln p}$. For example if $h(x) = \exp[\beta\cos(\alpha\ln x)]$, then it satisfies the above functional equation with $p = \exp(-2\pi)$ and $\psi(x)$ monotone increasing with $0 < \beta < 1$. The semi–Pareto distribution can be viewed as a more general class which includes the Pareto type III distribution when $\psi(x) = cx^{\alpha}$, where c is a constant.

Exercises 4.3.

4.3.1. Examine whether the following distributions are infinitely divisible.

(i) normal	(ii) exponential	(iii) Laplace	(iv) Cauchy
(v) binomial	(vi) Poisson	(vii) Geometric	(viii) negative binomial

4.3.2. Show that exponential distribution is geometric infinite divisible and self-decomposable.

4.3.3. Examine whether Cauchy distribution is self-decomposable.

4.3.4. Show that (i) Mittag-Leffler distribution is g.i.d. and belongs to class L. (ii) α-Laplace distribution is g.i.d. and self-decomposable.

4.3.5. Give a distribution which is infinitely divisible but not g.i.d.

4.3.6. Show that the AR(1) structure $x_n = ax_{n-1} + \epsilon_n$; $a \in (0, 1)$ is stationary Markovian if and only if $\{x_n\}$ is self-decomposable.

4.3.7. Obtain the stationary distribution of $\{\epsilon_n\}$ in the AR(1) structure $x_n = ax_{n-1} + \epsilon_n$; $a \in (0, 1)$ when $\{x_n\}$ follows exponential distribution. Generalize it to the case of Mittag-Leffler random variables.

4.3.8. Obtain the structure of the innovation distribution if $\{x_n\}$ follows α-Laplace distribution where $x_n = ax_{n-1} + \epsilon_n$. Deduce the case when $\alpha = 2$.

4.3.9. Show that if $\{x_n\}$ follows Cauchy distribution then $\{\epsilon_n\}$ also follows a Cauchy distribution in the AR(1) equation $x_n = ax_{n-1} + \epsilon_n$.

4.3.10. show that geometric and negative binomial distributions are discrete self-decomposable.

4.3.11. Consider the symmetric stable distribution with characteristic function $\varphi(t) = e^{-|t|^\alpha}$. Is it self-decomposable ?

4.4. Stationary Time Series

A time series, $\{x_t\}$, is a family of real–valued random variables indexed by $t \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers. More specifically, it is referred to as a discrete parameter time series. The time series $\{x_t\}$ is said to be stationary if, for any $t_1, t_2, \ldots$, $t_n \in \mathbb{Z}$, any $k \in \mathbb{Z}$, and $n = 1, 2, \ldots,$

$$F_{x_{t_1},x_{t_2},\ldots,x_{t_n}}(x_1, x_2, \ldots, x_n) = F_{x_{t_1+k},x_{t_2+k},\ldots,x_{t_n+k}}(x_1, x_2, \ldots, x_n)$$

where F denotes the distribution function of the set of random variables which appear as suffices. This is called stationarity in the strict sense.

Less stringently, we say a process $\{x_n\}$ is weakly stationary if the mean and variance of x_t remain constant over time and the covariance between any two values x_t and x_s depends only on the time difference and not on their individual time points.

$\{x_t\}$ is called a Gaussian process if, for all $t_n; n \geq 1$ the set of random variables $\{x_{t_1}, x_{t_2}, \ldots, x_{t_n}\}$ has a multivariate normal distribution.

Since a multivariate normal distribution is completely specified by its mean vector and covariance matrix, it follows that for a Gaussian process weak stationarity implies complete stationarity. But for non–Gaussian processes, this may not hold.

4.4.1. Autoregressive models

The era of linear time series models began with autoregressive models first introduced by Yule in 1927. The standard form of an autoregressive model of order p, denoted by AR(p), is given by

$$x_t = \sum_{j=1}^{p} a_j x_{t-j} + \epsilon_t; \qquad t = 0, \pm 1, \pm 2, \ldots$$

where $\{\epsilon_t\}$ are independent and identically distributed random variables called innovations and a_j, p are fixed parameters, with $a_p \neq 0$.

Another kind of model of great practical importance in the representation of observed time series is the moving average model. The standard form of a moving average model of order q, denoted by MA(q), is given by $x_t = \sum_{j=1}^{q} b_j \epsilon_{t-j} + \epsilon_t;\ t \in \mathbb{Z}$ where b_j, q are fixed parameters, with $b_q \neq 0$.

To achieve greater flexibility in the fitting of actually observed time series, it is more advantageous to include both autoregressive and moving average terms in the model. Such models called autoregressive–moving average models, denoted by ARMA(p,q), have the form

$$x_t = \sum_{j=1}^{p} a_j x_{t-j} + \sum_{k=1}^{q} b_k \epsilon_{t-k} + \epsilon_t; \qquad t \in \mathbb{Z}$$

where $\{a_j\}_{j=1}^{p}$ and $\{b_k\}_{k=1}^{q}$ are real constants called parameters of the model. It can be seen that an AR(p) model is the same as an ARMA(p,o) model and a MA(q) model is the same as an ARMA(o,q) model.

With the introduction of various non–Gaussian and non–linear models, the standard form of autoregression was widened in several respects.

A more general definition of autoregression of order p is given in terms of the linear conditional expectation requirement that

$$E(x_t | x_{t-1}, x_{t-2}, \ldots) = \sum_{j=1}^{p} a_j x_{t-j}$$

This definition could apply to models which are not of the linear form (see Lawrance (1991)).

4.4.2. A general solution

We consider a first order autoregressive model with innovation given by the structural relationship

$$x_n = \epsilon_n + \begin{cases} 0 & \text{with probability } p \\ x_{n-1} & \text{with probability } 1 - p \end{cases} \tag{4.4.1}$$

where $p \in (0, 1)$ and $\{\epsilon_n\}$ is a sequence of independent and identically distributed (i.i.d.) random variables selected in such a way that $\{x_n\}$ is stationary Markovian with a given marginal distribution function F.

Let $\phi_x(t) = E[e^{-tx}]$ be the Laplace–Stieltjes transform of x. Then (4.4.1) gives

$$\phi_{x_n}(t) = \phi_{\epsilon_n}(t)[p + (1-p)\phi_{x_{n-1}}(t)]$$

If we assume stationarity, this simplifies to

$$\phi_\epsilon(t) = \frac{\phi_x(t)}{p + (1-p)\phi_x(t)} \tag{4.4.2}$$

or equivalently

$$\phi_x(t) = \frac{p\phi_\epsilon(t)}{1 - (1-p)\phi_\epsilon(t)}. \tag{4.4.3}$$

When $\{x_n\}$ is marginally distributed as exponential, it is easy to see that (4.4.1) gives the TEAR(1) model.

We note that $\phi_\epsilon(t)$ in (4.4.2) does not represent a Laplace transform always. In order that the process given by (4.4.1) is properly defined, there should exist an innovation distribution such that $\phi_\epsilon(t)$ is a Laplace transform for all $p \in (0,1)$. To establish the main results we need the following lemmas.

Lemma 4.4.1. (Pillai (1990b)) Let F be a distribution with positive support and $\phi(t)$ be its Laplace transform. Then F is geometrically infinitely divisible if and only if

$$\phi(t) = \frac{1}{1 + \psi(t)}$$

where $\psi(t)$ is Bernstein with $\psi(0) = 0$.

Now we consider the following definition from Pillai (1990b).

Definition 4.4.1. For any non–vanishing Laplace transform $\phi(t)$, the function $\psi(t) = \frac{1}{\phi(t)} - 1$ is called the third characteristic.

Lemma 4.4.2. Let $\psi(t)$ be the third characteristic of $\phi(t)$. Then $p\psi(t)$ is a third characteristic for all $p \in (0,1)$ if and only if $\psi(t)$ has complete monotone derivative and $\psi(0) = 0$.

Thus we have the following theorem.

Theorem 4.4.1. *$\phi_\epsilon(t)$ in (4.4.2) represents a Laplace transform for all $p \in (0,1)$ if and only if $\phi_x(t)$ is the Laplace transform of a geometrically infinitely divisible distribution.*

This leads to the following theorem which brings out the role of geometrically infinitely divisible distributions in defining the new first order autoregressive model given by (4.4.1).

Theorem 4.4.2. *The innovation sequence* $\{\epsilon_n\}$ *defining the first order autoregressive model given by*

$$x_n = \epsilon_n + \begin{cases} 0 & \textit{with probability } p \\ x_{n-1} & \textit{with probability } 1-p \end{cases}$$

where $p \in (0,1)$, *exists if and only if the stationary marginal distribution of* x_n *is geometrically infinitely divisible. Then the innovation distribution is also geometrically infinitely divisible.*

Proof. Suppose that an innovation sequence $\{\epsilon_n\}$ such that the model (4.4.1) is properly defined exists. This implies that $\phi_\epsilon(t)$ in (4.4.2) is a Laplace transform for all $p \in (0,1)$. Then from (4.4.3)

$$\begin{aligned} \phi_x(t) &= p\phi_\epsilon(t)[1-(1-p)\phi_\epsilon(t)]^{-1} \\ &= \sum_{n=1}^{\infty} p(1-p)^{n-1}[\phi_\epsilon(t)]^n \end{aligned}$$

showing that the stationary marginal distribution of x_n is geometrically infinitely divisible. Conversely, if x_n has a stationary marginal distribution which is geometrically infinitely divisible, then $\phi_x(t) = \dfrac{1}{1+\psi(t)}$ where $\psi(t)$ has complete monotone derivative and $\psi(0) = 0$. Then from (4.4.2) we get $\phi_\epsilon(t) = \dfrac{1}{1+p\psi(t)}$, which establishes the existence of an innovation distribution, which is geometrically infinitely divisible.

4.4.3. Extension to a k-th order autoregressive model

In this section we consider an extension of the model given by (4.4.1) to the k-th order. The structure of this model is given by

$$x_n = \epsilon_n + \begin{cases} 0 & \text{with probability } p_0 \\ x_{n-1} & \text{with probability } p_1 \\ \vdots & \\ x_{n-k} & \text{with probability } p_k \end{cases} \tag{4.4.4}$$

where $p_i \in (0,1)$ for $i = 0, 1, \ldots, k$ and $p_0 + p_1 + \cdots + p_k = 1$. Taking Laplace transforms on both sides of (4.4.4) we get

$$\phi_{x_n}(t) = \phi_{\epsilon_n}(t)\left[p_0 + \sum_{i=1}^{k} p_i \phi_{x_{n-i}}(t)\right]$$

Assuming stationarity, it simplifies to

$$\begin{aligned}\phi_x(t) &= \phi_\epsilon(t)\left[p_0 + \sum_{i=1}^{k} p_i \phi_x(t)\right] \\ &= \phi_\epsilon(t)[p_0 + (1-p_0)\phi_x(t)].\end{aligned}$$

This yields

$$\phi_\epsilon(t) = \frac{\phi_x(t)}{p_0 + (1-p_0)\phi_x(t)} \tag{4.4.5}$$

which is analogous to the expression (4.4.2).

It may be noted that $k = 1$ corresponds to the first order model with $p = p_0$. From (4.4.6) it follows that the results obtained in Section 4.4.2 hold good for the k-th order model given by (4.4.5). This establishes the importance of geometrically infinitely divisible distributions in autoregressive modelling.

4.4.4. Mittag-Leffler autoregressive structure

The Mittag-Leffler distribution was introduced by Pillai (1990a) and has Laplace transform $\phi(t) = \frac{1}{1+t^\alpha}$, $0 < \alpha \le 1$. When $\alpha = 1$, this corresponds to the exponential distribution with unit mean. Jayakumar and Pillai (1993) considered the semi–Mittag-Leffler distribution with exponent α. Its Laplace transform is of the form $\frac{1}{1+\eta(t)}$ where $\eta(t)$ satisfies the functional equation

$$\eta(t) = a\eta(bt), \qquad 0 < b < 1 \tag{4.4.6}$$

and a is the unique solution of $ab^\alpha = 1$ where $0 < \alpha \le 1$. Then by Lemma 4.2.1 of Jayakumar and Pillai (1993), the solution of the functional equation (4.4.6) is $\eta(t) = t^\alpha h(t)$ where $h(t)$ is periodic in $\ln t$ with period $-\frac{2\pi\alpha}{\ln b}$. When $h(t) = 1$, $\eta(t) = t^\alpha$ and hence the Mittag-Leffler distribution is a special case of the semi-Mittag-Leffler distribution. It is obvious that the semi–Mittag-Leffler distribution is geometrically infinitely divisible.

Now we bring out the importance of the semi-Mittag-Leffler distribution in the context of the new autoregressive structure given by (4.4.1). The following theorem establishes this.

Theorem 4.4.3. *For a positive valued first order autoregressive process* $\{x_n\}$ *satisfying* (4.4.1) *the stationary marginal distribution of* x_n *and* ϵ_n *are identical except for a scale change if and only if* x_n*'s are marginally distributed as semi-Mittag-Leffler .*

Proof. Suppose that the stationary marginal distributions of x_n and ϵ_n are identical. This implies $\phi_\epsilon(t) = \phi_x(ct)$ where c is a constant. Then from (4.4.2) we get

$$\phi_x(ct) = \frac{\phi_x(t)}{p + (1-p)\phi_x(t)} \tag{4.4.7}$$

Writing $\phi_x(t) = \dfrac{1}{1+\eta(t)}$ in (4.4.7) we get

$$\frac{1}{1+\eta(ct)} = \frac{1}{1+p\eta(t)}$$

so that

$$\eta(ct) = p\eta(t)$$

By choosing $c = p^{1/\alpha}$, it follows that x_n is distributed as semi–Mittag-Leffler with exponent α.

Conversely, we assume that the stationary marginal distribution of x_n is semi–Mittag-Leffler . Then from (4.4.2)

$$\phi_\epsilon(t) = \frac{1}{1+p\eta(t)} = \frac{1}{1+\eta(p^{1/\alpha}t)}.$$

This establishes that $\epsilon_n \stackrel{d}{=} p^{1/\alpha} M_n$ where $\{M_n\}$ are independently and identically distributed as semi–Mittag-Leffler .

It can be easily seen that the above result is true in the case of the k-th order autoregressive model given by (4.4.4) also.

Exercises 4.4.

4.4.1. Consider a Poisson process $\{x(t)\}$ where $p[x(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}; n = 0, 1, \cdots$ Find $E(x(t))$ and $\text{Var}(x(t))$. Is the process stationary?

4.4.2. Consider a Poisson process $\{x(t)\}$ as above. Let x_0 be independent of $x(t)$ such that $p(x_0 = 1) = p(x_0 = -1) = \frac{1}{2}$. Define $N(t) = x_0(-1)^{N(t)}$. Find $E(N(t))$ and $\text{Cov}(N(t), \text{N(t+s)})$.

4.4.3. Define an AR(1) process and obtain the stationary solution for the distribution of $\{\epsilon_n\}$ when $\{x_n\}$ are exponentially distributed.

4.4.4. Show that an AR(1) model can be expressed as an $MA(\infty)$ model.

4.4.5. Construct a new AR(1) model with exponential innovations.

4.4.6. Examine whether the two-parameter Gamma distribution is g.i.d., giving conditions if any.
4.4.7. Show that exponential distribution is a special case of Mittag-Leffler distribution.

4.5. A Structural Relationship

In this section we obtain the specific structural relationship between the stationary marginal distributions of x_n and ϵ_n in the new autoregressive model.

Fujita (1993) generalized the results on Mittag-Leffler distributions and obtained a new characterization of geometrically infinitely divisible distributions with positive support using Bernstein functions. It was established that a distribution function G with $G(0) = 0$ is geometrically infinitely divisible if and only if G can be expressed in the form.

$$G(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\lambda^n W^{n*}([0, x]); \qquad x > 0,\ \lambda > 0 \tag{4.5.1}$$

where $W^{n*}(dx)$ is the n-fold convolution measure of a unique positive measure $W(dx)$ on $[0, \infty)$ such that

$$\frac{1}{f(x)} = \int_0^{\infty} e^{-sx} W(ds); \qquad x > 0 \tag{4.5.2}$$

for some Bernstein function f such that $\lim_{x\downarrow o} f(x) = 0$ and $\lim_{x\to\infty} f(x) = \infty$. Then the Laplace transform of $G(x)$ is $\dfrac{\lambda}{\lambda + f(t)}$. Using this result we get the following theorem.

Theorem 4.5.1. *The k-th order autoregressive equation given by (4.4.4) defines a stationary process with a given marginal distribution function $F_x(x)$ for x_n if and only if $F_x(x)$ can be expressed in the form*

$$F_x(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\lambda^n W^{n*}([0, x]); \qquad x > 0,\ \lambda > 0. \tag{4.5.3}$$

Then the innovations $\{\epsilon_n\}$ have a distribution function $F_\epsilon(x)$ given by

$$F_\epsilon(x) = \sum_{n=1}^{\infty}(-1)^{n+1}(\lambda/p_0)^n W^{n*}([0, x]); \qquad x > 0,\ \lambda > 0, \tag{4.5.4}$$

where $p_0 \in (0, 1)$ and W^{n*} is as in (4.5.1).

Proof. We have from Theorem 4.4.1 that $F_x(x)$ is geometrically infinitely divisible. Then (4.5.3) follows directly from Fujita (1993).

Now by substituting $\phi_x(t) = \dfrac{\lambda}{\lambda + f(t)}$ in (4.3.2) we get

$$\phi_\epsilon(t) = \frac{\lambda}{\lambda + p_0 f(t)} = \frac{(\lambda/p_0)}{(\lambda/p_0) + f(t)}$$

which leads to (4.5.4). This completes the proof.

The above theorem can be used to construct various autoregressive models under different stationary marginal distributions for x_n.

For example, the TEAR(1) model of Lawrance and Lewis (1981) can be obtained by taking $f(t) = t$. Then $W^{n*}([0,x]) = \dfrac{x^n}{n!}$ so that $F_x(x) = 1 - e^{-\lambda x}$ and $F_\epsilon(x) = 1 - e^{-(\lambda/p)x}$. If we take $\lambda = 1$ and $f(t) = t^\alpha$; $0 < \alpha \leq 1$ we can obtain an easily tractable first order autoregressive Mittag-Leffler process denoted by TMLAR(1). In this case $W^{n*}([0,x]) = \dfrac{x^{n\alpha}}{\Gamma(1+n\alpha)}$. In a similar manner by taking $\lambda = 1$ and $f(t)$ satisfying the functional equation $f(t) = af(bt)$ where $a = b^{-\alpha}$; $0 < b < 1$, $0 < \alpha \leq 1$, we can obtain an easily tractable first order autoregressive semi–Mittag-Leffler process denoted by TSMLAR(1).

4.5.1. The TMLAR(1) process

An easily tractable form of a first order autoregressive Mittag-Leffler process, called TMLAR(1), is constituted by $\{x_n\}$ having a structure of the form

$$x_n = p^{1/\alpha} M_n + \begin{cases} 0 & \text{with probability p} \\ x_{n-1} & \text{with probability 1-p} \end{cases} \tag{4.5.5}$$

where $p \in (0,1)$; $0 < \alpha \leq 1$ and $\{M_n\}$ is independently and identically distributed as Mittag-Leffler with exponent α and $x_0 \stackrel{d}{=} M_1$. The model (4.5.5) can be rewritten in the form

$$x_n = p^{1/\alpha} M_n + I_n x_{n-1} \tag{4.5.6}$$

where $\{I_n\}$ is a Bernoulli sequence such that $P(I_n = 0) = p$ and $P(I_n = 1) = 1 - p$.

If in the structural form (4.5.5), we assume that $\{M_n\}$ are distributed as semi–Mittag-Leffler with exponent α, then $\{x_n\}$ constitute a tractable semi–Mittag-Leffler autoregressive process of order 1, called TSMLAR(1). Both models are Markovian and stationary. It can be seen that the TMLAR(1) process is a special case of the TSMLAR(1) process since the Mittag-Leffler distribution is a special case of the semi–Mittag-Leffler distribution.

Now we shall consider the TSMLAR(1) process and establish that it is strictly stationary and Markovian, provided x_0 is distributed as semi–Mittag-Leffler . In order to prove this we use the method of induction.

Suppose that x_{n-1} is distributed as semi–Mittag-Leffler (α). Then by taking Laplace transforms on both sides of (4.5.5), we get

$$\begin{aligned}\phi_{x_n}(t) &= \phi_{M_n}(p^{1/\alpha}t)[p + (1-p)\phi_{x_{n-1}}(t)] \\ &= \frac{1}{1+\eta(p^{1/\alpha}t)}\left[p + (1-p)\frac{1}{1+\eta(t)}\right] \\ &= \frac{1}{1+p\eta(t)} \cdot \frac{1+p\eta(t)}{1+\eta(t)} \\ &= \frac{1}{1+\eta(t)}.\end{aligned}$$

Hence x_n is distributed as semi–Mittag-Leffler with exponent α.

If x_0 is arbitrary, then also it is easy to establish that $\{x_n\}$ is asymptotically stationary. Thus we have the following theorem.

Theorem 4.5.2. *The first order autoregressive equation*

$$x_n = p^{1/\alpha}M_n + I_n x_{n-1}; \qquad n = 1, 2, \ldots, \quad p \in (0, 1)$$

where $\{I_n\}$ are independent Bernoulli random variables such that $P(I_n = 0) = p = 1 - P(I_n = 1)$ defines a positive valued strictly stationary first order autoregressive process if and only if $\{M_n\}$ are independently and identically distributed as semi–Mittag-Leffler with exponent α and $x_0 \overset{d}{=} M_1$.

Remark If we consider characteristic functions instead of Laplace transforms, the results can be applied to real valued autoregressive processes. Then the role of semi–Mittag-Leffler distributions is played by semi–α–Laplace distributions introduced by Pillai (1985).

4.5.2. The NEAR(1) model

In this section we consider a generalized form of the first order autoregressive equation. The new structure is given by

$$x_n = \epsilon_n + \begin{cases} 0 & \text{with probability } p \\ a x_{n-1} & \text{with probability } 1-p \end{cases} \tag{4.5.7}$$

where $0 \le p \le 1$; $0 \le a \le 1$ and $\{\epsilon_n\}$ is a sequence of independent and identically distributed random variables such that $\{x_n\}$ have a given stationary marginal distribution. Let $\phi_x(t) = E[\mathrm{e}^{-tx}]$ be the Laplace–Stieltjes transform of x. Then (4.5.7) gives

$$\phi_{x_n}(t) = \phi_{\epsilon_n}(t)[p + (1-p)\phi_{x_{n-1}}(at)]$$

Assuming stationarity, it simplifies to

$$\phi_\epsilon(t) = \frac{\phi_x(t)}{p + (1-p)\phi_x(at)}. \tag{4.5.8}$$

When $p = 0$ and $0 < a < 1$, the model (4.5.7) is the standard first order autoregressive model. Then the model is properly defined if and only if the stationary marginal distribution of x_n is self–decomposable. When $a = 1$, $0 < p < 1$ the model is the same as the model (4.4.1), which is properly defined if and only if the stationary marginal distribution of x_n is geometrically infinitely divisible. When $a = 0$ or $p = 1$, x_n and ϵ_n are identically distributed.

Now we consider the case when $a \in (0,1]$ and $p \in (0,1]$, but not simultaneously equal to 1. Lawrance and Lewis (1981) developed an NEAR(1) model with exponential (λ) marginal distribution for x_n. Then $\phi_x(t) = \dfrac{\lambda}{\lambda + t}$ and substitution in (4.5.8) gives

$$\phi_\epsilon(t) = \frac{\lambda + at}{\lambda + t} \cdot \frac{\lambda}{\lambda + pat} \tag{4.5.9}$$

which can be rewritten as

$$\phi_\epsilon(t) = \left(\frac{1-a}{1-pa}\right)\left(\frac{\lambda}{\lambda+t}\right) + \left[\frac{(1-p)a}{1-pa}\right]\left(\frac{\lambda}{\lambda+pat}\right).$$

Hence ϵ_n can be regarded as a convex exponential mixture of the form

$$\epsilon_n = \begin{cases} E_n & \text{with probability } \frac{1-a}{1-pa} \\ paE_n & \text{with probability } \frac{(1-p)a}{1-pa} \end{cases} \tag{4.5.10}$$

where $\{E_n\}$; $n = 1, 2, \ldots$ are independent and identically distributed as exponential (λ) random variables. Another representation for ϵ_n can be obtained from (4.5.9) by writing

$$\phi_\epsilon(t) = \left[a + (1-a)\frac{\lambda}{\lambda+t}\right]\left[\frac{\lambda}{\lambda+pat}\right]. \tag{4.5.11}$$

Then writing w.p. for 'with probability' ϵ_n can be regarded as the sum of two independent random variables u_n and v_n where

$$u_n = \begin{cases} 0 & \text{w. p. } a \\ E_n & \text{w. p. } 1-a \text{ and} \end{cases} \qquad v_n = paE_n \tag{4.5.12}$$

where $\{E_n\}$; $n = 1, 2, \ldots$ are exponential (λ). It may be noted that when $p = 0$, the model is identical with the EAR(1) process, of Gaver and Lewis (1980). Thus the new representation of ϵ_n seems to be more appropriate, when NEAR(1) process is regarded as a generalization of the EAR(1) process.

Exercises 4.5.

4.5.1. If $f(t) = t$, find $W^{n^*}([0, x])$.

4.5.2. If $f(t) = t^\alpha$, find $F_\epsilon(x)$.

4.5.3. State any three distributions belonging to the semi-Mittag-Leffler family.

4.5.4. Show that the stationary solution of Equation 4.5.7 is a family consisting of g.i.d. and class L distributions.

4.5.5. Obtain the innovation structure of the NEAR(1) model.

4.5.6. Obtain the innovation structure of the NMLAR(1) model.

4.6. New Mittag-Leffler Autoregressive Models

Now we construct a new first order autoregressive process with Mittag-Leffler marginal distribution, called the NMLAR(1) model. The structure of the model is as in (4.5.7) and the innovations can be derived by substituting $\phi_X(t) = \dfrac{1}{1 + t^\alpha}$; $0 < \alpha \le 1$ in (4.5.8). This gives

$$\phi_\epsilon(t) = \frac{1 + a^\alpha t^\alpha}{1 + t^\alpha} \cdot \frac{1}{1 + a^\alpha p t^\alpha}.$$

Hence the innovations ϵ_n can be given in the form

$$\epsilon_n = \begin{cases} M_n & \text{with probability } \frac{1-a^\alpha}{1-pa^\alpha} \\ pa^\alpha M_n & \text{with probability } \frac{(1-p)a^\alpha}{1-pa^\alpha} \end{cases} \tag{4.6.1}$$

where $\{M_n\}$ are Mittag-Leffler (α) random variables.

An alternate representation of ϵ_n is $\epsilon_n = u_n + v_n$ where u_n and v_n are independent random variables such that

$$u_n = \begin{cases} 0 & \text{w.p. } a^\alpha \\ M_n & \text{w.p. } 1 - a^\alpha \end{cases} \text{ and } v_n = ap^{1/\alpha} M_n \tag{4.6.2}$$

where $\{M_n\}$; $n = 1, 2, \ldots$ are independent Mittag-Leffler (α) random variables.

It can be shown that the process is strictly stationary and Markovian. This gives us the following theorem.

Theorem 4.6.1. *The first order autoregressive equation given by* (4.5.7) *defines a strictly stationary* AR(1) *process with a Mittag-Leffler* (α) *marginal distribution for x_n if and only if the innovations are of the form $\epsilon_n = u_n + v_n$ where u_n and v_n are as in* (4.6.2) *with x_0 distributed as Mittag-Leffler* (α).

Proof. We prove this by induction. We assume that x_{n-1} is Mittag-Leffler (α). Then by taking Laplace transforms, we get

$$\begin{aligned}
\phi_{x_n}(t) &= \phi_{M_n}(ap^{1/\alpha}t) \cdot [a^\alpha + (1-a^\alpha)\phi_{M_n}(t)] \\
&\times [p + (1-p)\phi_{x_{n-1}}(at)] \\
&= \frac{1}{1 + a^\alpha p t^\alpha} \cdot \left[a^\alpha + (1-a^\alpha)\frac{1}{1+t^\alpha}\right] \\
&\times \left[p + (1-p)\frac{1}{1+a^\alpha t^\alpha}\right] \\
&= \frac{1}{1+a^\alpha p t^\alpha} \cdot \frac{1+a^\alpha t^\alpha}{1+t^\alpha} \cdot \frac{1+pa^\alpha t^\alpha}{1+a^\alpha t^\alpha} \\
&= \frac{1}{1+t^\alpha}.
\end{aligned}$$

This shows that x_n is distributed as Mittag-Leffler (α), and this establishes the sufficiency part.

The necessary part is obvious from the derivation of the innovation sequence. This completes the proof.

The joint distribution of (x_n, x_{n-1}) is of interest in describing the process and matching it with data. Therefore, we shall obtain the joint distribution with the use of Laplace-Stieltjes transforms. The bivariate Laplace transform is given by

$$\begin{aligned}
\phi_{x_n, x_{n-1}}(s,t) &= E\{\exp(-sx_n - tx_{n-1})\} \\
&= \phi_\epsilon(s)\{p\phi_x(t) + (1-p)\phi_x(as+t)\} \\
&= \frac{1+a^\alpha s^\alpha}{1+s^\alpha} \cdot \frac{1}{1+pa^\alpha s^\alpha} \cdot \left\{\frac{p}{1+t^\alpha} + \frac{1-p}{1+(as+t)^\alpha}\right\}.
\end{aligned}$$

It is possible to obtain the joint distribution by inverting this expression.

4.6.1. The NSMLAR(1) process

Now we extend the NMLAR(1) process to a wider class to construct a new semi–Mittag-Leffler first order autoregressive process. The process has the structure

$$x_n = \epsilon_n + \begin{cases} 0 & \text{with probability } p \\ a x_{n-1} & \text{with probability } 1-p \end{cases}$$

where $\{\epsilon_n\}$ are independently and identically distributed as the sum of two independent random variables u_n and v_n where

$$u_n = \begin{cases} 0 & \text{w.p. } a^\alpha \\ M_n & \text{w.p. } 1-a^\alpha \end{cases} \text{ and } v_n = ap^{1/\alpha} M_n \tag{4.6.3}$$

where $\{M_n\}$; $n = 1, 2, \ldots$ are independently and identically distributed as semi–Mittag-Leffler (α).

This process is also clearly strictly stationary and Markovian provided x_0 is semi–Mittag-Leffler (α). This follows by induction. In terms of Laplace transforms we have

$$\begin{aligned}
\phi_{x_n}(t) &= [p + (1-p)\phi_{x_{n-1}}(at)][a^\alpha + (1-a^\alpha)\phi_{M_n}(t)] \\
&\quad \cdot [\phi_{M_n}(ap^{1/\alpha}t)] \\
&= \left[p + (1-p)\cdot\frac{1}{1+\eta(at)}\right]\left[a^\alpha + (1-a^\alpha)\cdot\frac{1}{1+\eta(t)}\right] \\
&\quad \cdot \frac{1}{1+\eta(ap^{1/\alpha}t)} \\
&= \left[p + (1-p)\frac{1}{1+a^\alpha\eta(t)}\right]\cdot\left[\frac{1+a^\alpha\eta(t)}{1+\eta(t)}\right] \\
&\quad \cdot \frac{1}{1+a^\alpha p\eta(t)} \\
&= \frac{1}{1+\eta(t)}.
\end{aligned}$$

Thus we have established the following theorem.

Theorem 4.6.2. *The first order autoregressive equation*

$$x_n = aI_n x_{n-1} + \epsilon_n; \quad n = 1, 2, \ldots$$

where $\{I_n\}$ are independent Bernoulli sequences such that $P(I_n = 0) = p$ and $P(I_n = 1) = 1-p$; $p \in (0,1)$, $a \in (0,1)$ is a strictly stationary AR(1) process with semi–Mittag-Leffler (α) marginal distribution if and only if $\{\epsilon_n\}$ are independently and *identically distributed as the sum of two independent random variables* u_n *and* v_n *as in* (3.8.1) *and* x_0 *is distributed as semi–Mittag-Leffler* (α)*.*

When $\eta(t) = t^{\alpha}$, the NSMLAR(1) model becomes the NMLAR(1) model.

4.6.2. Semi–Mittag-Leffler tailed processes

In an attempt to develop autoregressive models for time series with exact zeroes Littlejohn (1993) formulated an autoregressive process with exponential tailed marginal distribution, after the new exponential autoregressive process (NEAR(1)) of Lawrance and Lewis (1981). However, the primary aim of Littlejohn was to extend the time reversibility theorem of Chernick *et al.*(1988) and hence the model was not studied in detail. Hence we intend to make a detailed study on this process. Here the tail of a non–negative random variable refers to the positive part of the sample space, excluding only the point zero.

Definition 4.6.1. A random variable E is said to have the exponential tailed distribution denoted by $ET(\lambda, \theta)$ if $P(E = 0) = \theta$ and $P(E > x) = (1-\theta)e^{-\lambda x}$; $x > 0$ where $\lambda > 0$ and $0 \le \theta < 1$. Then the Laplace-Stieltjes transform of E is given by

$$\begin{aligned}\phi_E(t) &= \theta + (1-\theta)\frac{\lambda}{\lambda+t}\\ &= \frac{\lambda+\theta t}{\lambda+t}\end{aligned}$$

4.6.3. The exponential tailed autoregressive process [ETAR(1)]

It is evident that the exponential tailed distribution is not self–decomposable and so it cannot be marginal to the autoregressive structure of Gaver and Lewis (1980). But an autoregressive process satisfying the NEAR(1) structure given by (4.5.7) can be constructed as follows.

We have from (4.5.8), by substituting $\phi_x(t) = \frac{\lambda+\theta t}{\lambda+t}$, the Laplace transform of the innovation ϵ_n in the stationary case as

$$\begin{aligned}\phi_\epsilon(t) &= \left[\frac{\lambda+\theta t}{\lambda+t}\right]\left[\frac{\lambda+at}{\lambda+a[p+(1-p)\theta]t}\right]\\ &= \left[\frac{\lambda+at}{\lambda+t}\right]\left[\frac{\lambda+\theta t}{\lambda+bt}\right]\end{aligned}$$

where $b = a[p + (1 - p)\theta]$

$$\phi_\epsilon(t) = \left[a + (1-a)\frac{\lambda}{\lambda + t}\right]\left[\frac{\theta}{b} + \left(1 - \frac{\theta}{b}\right)\frac{(\lambda/b)}{(\lambda/b) + t}\right]$$

so that the innovations $\{\epsilon_n\}$ can be represented as the sum of two independent exponential tailed random variables u_n and v_n where

$$u_n \overset{d}{=} ET(\lambda, a) \qquad \text{and} \;\; v_n \overset{d}{=} ET\left(\lambda', \theta'\right) \tag{4.6.4}$$

where $\lambda' = \lambda/b$ and $\theta' = \theta/b$, provided $\theta \leq b$. Since $p \leq 1$, we require that $\theta \leq a$. Thus the ETAR(1) process can be defined as a sequence $\{x_n\}$ satisfying (4.5.3) where $\{\epsilon_n\}$ is a sequence of independent and identically distributed random variables such that $\epsilon_n = u_n + v_n$ where u_n and v_n are as in (4.6.4).

It can be easily shown that the process is strictly stationary and Markovian provided x_0 is distributed as $ET(\lambda, \theta)$. This follows by mathematical induction since

$$\begin{aligned}
\phi_{x_n}(t) &= \phi_{\epsilon_n}(t) \cdot [p + (1-p)\phi_{x_{n-1}}(at)] \\
&= \frac{\lambda + at}{\lambda + t} \cdot \frac{\lambda + \theta t}{\lambda + bt} \cdot \left[p + (1-p)\left(\frac{\lambda + \theta a t}{\lambda + at}\right)\right] \\
&= \frac{\lambda + at}{\lambda + t} \cdot \frac{\lambda + \theta t}{\lambda + bt} \cdot \frac{\lambda + bt}{\lambda + at} \\
&= \frac{\lambda + \theta t}{\lambda + t}.
\end{aligned}$$

When $\theta = 0$, the $ET(\lambda, \theta)$ distribution reduces to the exponential (λ) distribution and the ETAR(1) model then becomes the NEAR(1) model.

4.6.4. The Mittag-Leffler tailed autoregressive process [MLTAR(1)]

The Mittag-Leffler tailed distribution has Laplace transform given by

$$\begin{aligned}
\phi_x(t) &= \theta + (1-\theta) \cdot \frac{1}{1 + t^\alpha} \\
&= \frac{1 + \theta t^\alpha}{1 + t^\alpha}; \qquad 0 < \alpha \leq 1
\end{aligned}$$

and the distribution shall be denoted by MLT (α, θ). Similarly for a two-parameter Mittag-Leffler random variable ML(α, λ) the Laplace transform of the tailed Mittag-Leffler distribution is given by $\phi_x(t) = \theta + (1-\theta)\frac{\lambda^\alpha}{\lambda^\alpha + t^\alpha} = \frac{\lambda^\alpha + \theta t^\alpha}{\lambda^\alpha + t^\alpha}$. This shall be denoted by MLT$(\alpha, \lambda, \theta)$.

The MLTAR(1) process has the general structure given by the equation (4.5.7). The innovation structure can be derived as follows.

$$\begin{aligned}\phi_\epsilon(t) &= \frac{1+\theta t^\alpha}{1+t^\alpha}\cdot\frac{1+a^\alpha t^\alpha}{1+a^\alpha[p+(1-p)\theta]t^\alpha}\\ &= \frac{1+a^\alpha t^\alpha}{1+t^\alpha}\cdot\frac{1+\theta t^\alpha}{1+ct^\alpha}\end{aligned}$$

where $c = a^\alpha[p+(1-p)\theta]$. Therefore

$$\phi_\epsilon(t) = \left[a^\alpha + (1-a^\alpha)\frac{1}{1+t^\alpha}\right]\left[\frac{\frac{1}{c}+\frac{\theta}{c}t^\alpha}{\frac{1}{c}+t^\alpha}\right].$$

Hence the innovation $\{\epsilon_n\}$ can be viewed as the sum of two independently distributed random variables u_n and v_n where

$$u_n \stackrel{d}{=} MLT(\alpha, a^\alpha)$$

and

$$v_n \stackrel{d}{=} MLT\,(\alpha, \lambda', \theta')$$

where $\lambda' = 1/c^{1/\alpha}$ and $\theta' = \theta/c$ provided $\theta \le c$. This holds when $\theta \le a^\alpha$.

The model can be extended to the class of semi–Mittag-Leffler distributions. Here we consider a semi–Mittag-Leffler distribution with Laplace transform

$$\phi_x(t) = \frac{\lambda^\alpha}{\lambda^\alpha + \eta(t)}$$

where $\eta(t)$ satisfies the functional equation

$$\eta(mt) = m^\alpha\eta(t); \qquad 0 < m < 1; \quad 0 < \alpha \le 1.$$

This is denoted by SML(α, λ). Then the semi–Mittag-Leffler tailed distribution denoted by SMLT$(\alpha, \lambda, \theta)$ has Laplace transform

$$\phi_x(t) = \frac{\lambda^\alpha + \theta\eta(t)}{\lambda^\alpha + \eta(t)}.$$

The first order semi-Mittag-Leffler tailed autoregressive (SMLTAR(1)) process has innovations whose Laplace transform is given by

$$\phi_\epsilon(t) = \left[\frac{\lambda^\alpha + \theta\eta(t)}{\lambda^\alpha + \eta(t)}\right]\left[\frac{\lambda^\alpha + a^\alpha\eta(t)}{\lambda^\alpha + c\eta(t)}\right]$$

where $c = a^\alpha[p + (1-p)\theta]$. Therefore

$$\begin{aligned}\phi_\epsilon(t) &= \left[\frac{\lambda^\alpha + a^\alpha\eta(t)}{\lambda^\alpha + \eta(t)}\right]\left[\frac{\lambda^\alpha + \theta\eta(t)}{\lambda^\alpha + c\eta(t)}\right]\\ &= \left[a^\alpha + (1-a^\alpha)\frac{\lambda^\alpha}{\lambda^\alpha + \eta(t)}\right]\left[\frac{\theta}{c} + \left(1 - \frac{\theta}{c}\right)\frac{\lambda^\alpha/c}{\lambda^\alpha/c + \eta(t)}\right].\end{aligned}$$

Therefore, the innovations $\{\epsilon_n\}$ can be represented as the sum of two independent semi–Mittag-Leffler tailed random variables u_n and v_n where

$$u_n \stackrel{d}{=} SMLT(\alpha, \lambda, a^\alpha) \text{ and } v_n \stackrel{d}{=} SMLT\,(\alpha, \lambda', \theta') \tag{4.6.5}$$

where $\lambda' = \lambda/c^{1/\alpha}, \theta' = \theta/c$. Then we have the following theorem which gives the stationary solution of the SMLTAR(1) model.

Theorem 4.6.3. *For $0 < p < 1$, $0 < a < 1$ the stationary Markov process $\{x_n\}$ defined by* (4.5.7) *has a semi–Mittag-Leffler tailed SMLT$(\alpha, \lambda, \theta)$ marginal distribution if and only if the innovation sequence $\{\epsilon_n\}$ are independent and identically distributed as the sum of two independent semi–Mittag-Leffler Tailed random variables as in* (4.6.5)*, provided $x_0 \stackrel{d}{=} SMLT(\alpha, \lambda, \theta)$.*

The stationarity of the process can be easily established, as given below.

$$\begin{aligned}\phi_{x_n}(t) &= \phi_{\epsilon_n}(t)[p + (1-p)\phi_{x_{n-1}}(at)]\\ &= \left[\frac{\lambda^\alpha + a^\alpha\eta(t)}{\lambda^\alpha + \eta(t)}\right]\left[\frac{\lambda^\alpha + \theta\eta(t)}{\lambda^\alpha + c\eta(t)}\right]\\ &\times\left[p + (1-p)\frac{\lambda^\alpha + \theta\eta(at)}{\lambda^\alpha + \eta(at)}\right]\\ &= \left[\frac{\lambda^\alpha + a^\alpha\eta(t)}{\lambda^\alpha + \eta(t)}\right]\left[\frac{\lambda^\alpha + \theta\eta(t)}{\lambda^\alpha + c\eta(t)}\right]\left[\frac{\lambda^\alpha + c\eta(t)}{\lambda^\alpha + \eta(at)}\right]\\ &= \frac{\lambda^\alpha + \theta\eta(t)}{\lambda^\alpha + \eta(t)} \quad \text{since } \eta(at) = a^\alpha\eta(t).\end{aligned}$$

Hence x_n is distributed as SMLT$(\alpha, \lambda, \theta)$. The necessity part follows easily from the derivation of the structure of the innovation sequence. Now we consider the following theorem.

Theorem 4.6.4. *In a positive valued stationary Markov process $\{x_n\}$ satisfying the first order autoregressive equation $x_n = ax_{n-1} + \epsilon_n$, $0 < a < 1$ the innovations $\{\epsilon_n\}$ are independently and identically distributed as a tailed distribution of the same type as that of $\{x_n\}$ if and only if $\{x_n\}$ are distributed as semi–Mittag-Leffler .*

Proof We have, assuming stationarity,

$$\phi_x(t) = \phi_x(at)\phi_\epsilon(t).$$

Suppose

$$\phi_\epsilon(t) = \theta + (1-\theta)\phi_x(t) \quad \text{where } 0 \le \theta < 1.$$

Then

$$\phi_x(t) = \phi_x(at)[\theta + (1-\theta)\phi_x(t)].$$

Writing

$$\phi_x(t) = \frac{1}{1+\eta(t)}, \qquad \text{we get}$$

$$\begin{aligned} \frac{1}{1+\eta(t)} &= \frac{1}{1+\eta(at)}\left[\theta + (1-\theta)\frac{1}{1+\eta(t)}\right] \\ &= \left[\frac{1}{1+\eta(at)}\right]\left[\frac{1+\theta\eta(t)}{1+\eta(t)}\right]. \end{aligned}$$

This implies $\eta(at) = \theta\eta(t)$. By taking $\theta = a^\alpha$, this means that the distribution of x_n is semi–Mittag-Leffler .

Conversely, if $\{x_n\}$ are semi–Mittag-Leffler , we get

$$\begin{aligned} \phi_\epsilon(t) = \frac{\phi_x(t)}{\phi_x(at)} &= \frac{1+\eta(at)}{1+\eta(t)} \\ &= \frac{1+a^\alpha\eta(t)}{1+\eta(t)} \\ &= a^\alpha + (1-a^\alpha)\frac{1}{1+\eta(t)}. \end{aligned}$$

Hence $\{\epsilon_n\}$ is distributed as SMLT(α, a^α).

The SMLTAR(1) process can be regarded as generalizations of the EAR(1), NEAR(1), MLAR(1), NMLAR(1), TEAR(1), ETAR(1) and MLTAR(1) processes. These processes are useful to model non-negative time series data which exhibit zeros, as in the case of stream flow data of rivers that are dry during part of the year. They are useful for modelling life times of devices which have some probability for damage immediately when it is put to use.

Now we extend our results to the case of real valued processes, in which case the time series data can be negative as well as positive, including zeros.

4.6.5. Semi–α–Laplace tailed first order autoregressive process [α–SLTAR(1)]

In this section we consider continuous random variables defined over $(-\infty, \infty)$ and construct a new random variable called a tailed random variable with an atom of mass θ at zero and the rest of the probability distributed over the sample space of the original random variable excluding the point zero. If the characteristic function of the original random variable is $\varphi(t)$ then the characteristic function of the new tailed random variable is given by $\theta + (1-\theta)\varphi(t)$.

Here we construct a time series model with stationary marginal distribution as a semi–α–Laplace tailed distribution with characteristic function $\theta + (1-\theta)\dfrac{1}{1+\eta(t)}$ where $\beta^\alpha \eta(t) = \eta(\beta t)$; $0 < \beta < 1$, $0 < \alpha \le 2$. This distribution shall be denoted by SALT(α, θ) and the corresponding first order semi–α–Laplace tailed autoregressive process by α-SLTAR(1).

The structure of the model is given by (4.5.7). The characteristic function of the innovation sequence $\{\epsilon_n\}$, under stationarity, is given by

$$\begin{aligned}\varphi_\epsilon(t) &= \frac{\varphi_x(t)}{p + (1-p)\varphi_x(at)}\\ &= \left[\frac{1+\theta\eta(t)}{1+\eta(t)}\right]\left[\frac{1+\eta(at)}{1+c\eta(t)}\right]\end{aligned}$$

where $c = a^\alpha[p + (1-p)\theta]$. Therfore

$$\begin{aligned}\varphi_\epsilon(t) &= \left(\frac{1+a^\alpha\eta(t)}{1+\eta(t)}\right)\left(\frac{\frac{1}{c}+\frac{\theta}{c}\eta(t)}{\frac{1}{c}+\eta(t)}\right)\\ &= \left[a^\alpha + (1-a^\alpha)\frac{1}{1+\eta(t)}\right]\left[\frac{\theta}{c} + \left(1-\frac{\theta}{c}\right)\cdot\frac{\frac{1}{c}}{\frac{1}{c}+\eta(t)}\right]\\ &= \left[a^\alpha + (1-a^\alpha)\frac{1}{1+\eta(t)}\right]\left[\frac{\theta}{c} + \left(1-\frac{\theta}{c}\right)\frac{1}{1+\eta(c^{1/\alpha}t)}\right].\end{aligned}$$

Therefore, ϵ_n can be regarded as the sum of two independent semi–α–Laplace tailed random variables u_n and v_n such that

$$u_n \stackrel{d}{=} SALT(\alpha, a^\alpha) \text{ and } v_n \stackrel{d}{=} c^{1/\alpha} SALT(\alpha, \theta/c). \tag{4.6.6}$$

The model is strictly stationary and Markovian. We have the following theorem.

Theorem 4.6.5. *For $0 < p < 1$, $0 < a < 1$ the stationary Markov process $\{x_n\}$ defined by (4.5.7) has a semi–α–Laplace tailed marginal distribution if and only if* *the innovation sequence* $\{\epsilon_n\}$ *are independent and identically distributed as the sum of two independent semi–α–Laplace tailed random variables having structure given by* (4.6.6), *provided* $x_0 \stackrel{d}{=} SALT(\alpha, \theta)$.

4.6.6. A first order α–Laplace tailed autoregressive [α-LTAR(1)] process

It can be seen that an α–Laplace tailed first order autoregressive process having structure (4.5.7) can be constructed for any a such that $|a| \leq 1$. In this case, the characteristic function of x_n is

$$\varphi_{x_n}(t) = \theta + (1-\theta)\frac{1}{1+|t|^\alpha} = \frac{1+\theta|t|^\alpha}{1+|t|^\alpha}; 0 < \alpha \leq 2.$$

Then (4.5.8) can be given in terms of characteristic function as

$$\varphi_\epsilon(t) = \left[\frac{1+\theta|t|^\alpha}{1+|t|^\alpha}\right]\left[\frac{1+|at|^\alpha}{1+c|t|^\alpha}\right]$$

where $c = |a|^\alpha[p + (1-p)\theta]$; $0 < \alpha \leq 2$. Therefore $\varphi_\epsilon(t)$ can be written as

$$\varphi_\epsilon(t) = \left\{|a|^\alpha + (1-|a|^\alpha)\cdot\frac{1}{1+|t|^\alpha}\right\}\left\{\frac{\theta}{c} + \left(1-\frac{\theta}{c}\right)\frac{1}{1+c|t|^\alpha}\right\}.$$

This shows that the innovations can be written as the sum of two independent α–Laplace tailed random variables of which one has an atom of mass $|a|^\alpha$ at zero and the other has an atom of mass $\frac{\theta}{c}$ at zero. The corresponding model may be called an α–Laplace tailed first order autoregressive process denoted by α–LTAR(1). The model is defined when $\theta \leq |a|^\alpha$; $0 < \alpha \leq 2$, $|a| < 1$. When $\theta = 0$, we get

$$\varphi_\epsilon(t) = \left[|a|^\alpha + (1-|a|^\alpha)\frac{1}{1+|t|^\alpha}\right]\left[\frac{1}{1+c|t|^\alpha}\right].$$

This gives a new α–Laplace autoregressive process with innovations of the form $\epsilon_n = u_n + v_n$ where u_n and v_n are indpendently distributed as

$$u_n = \begin{cases} 0 & \text{w.p. } |a|^\alpha \\ L_n & \text{w.p. } 1-|a|^\alpha \end{cases} \text{ and } v_n = c^{1/\alpha}L_n$$

where $\{L_n\}$ are α–Laplace variables. This model may be called a new α–Laplace autoregressive process of order one denoted by α–NLAR(1).

When $p = 0$, this model gives an α–Laplace autoregressive process denoted by α–LAR(1) with the usual linear additive structure. Then the model is given by

$$\begin{aligned} x_n &= a x_{n-1} + \epsilon_n \\ &= a x_{n-1} + \begin{cases} 0 & \text{with probability } |a|^\alpha \\ L_n & \text{with probability } 1 - |a|^\alpha \end{cases} \end{aligned}$$

provided $|a| \leq 1$; $0 < \alpha \leq 2$. Now we have the following theorem:

Theorem 4.6.6. *The first order difference equation*

$$x_n = a x_{n-1} + \begin{cases} 0 & \textit{with probability } \; |a|^\alpha \\ L_n & \textit{with probability } \; 1 - |a|^\alpha \end{cases} \tag{4.6.7}$$

where $|a| < 1$, $0 < \alpha \leq 2$ defines a strictly stationary Markovian first order autoregressive process with α–Laplace marginal distribution, if and only if the innovations $\{L_n\}$ are independent and identically distributed as α–Laplace and are independent of $x_0 \stackrel{d}{=} \alpha$–Laplace.

Proof. Assuming that the characteristic function of x_n is denoted by $\varphi_{x_n}(t)$, we have from (4.6.7)

$$\varphi_{x_n}(t) = \varphi_{x_{n-1}}(at)\{|a|^\alpha + (1 - |a|^\alpha)\varphi_{L_n}(t)\}$$

Assuming stationarity

$$\varphi_x(t) = \varphi_x(at)\{|a|^\alpha + (1 - |a|^\alpha)\varphi_L(t)\}$$

Therefore,

$$\varphi_L(t) = \frac{\varphi_x(t) - |a|^\alpha \varphi_x(at)}{(1 - |a|^\alpha)\varphi_x(at)}.$$

Substituting $\varphi_x(t) = \dfrac{1}{1 + |t|^\alpha}$ we get

$$\varphi_L(t) = \frac{1}{1 + |t|^\alpha}.$$

This establishes the necessary part. The converse is proved by mathematical induction. Assume that $x_{n-1} \stackrel{d}{=} \alpha$–Laplace. Then

$$\begin{aligned}\varphi_{x_n}(t) &= \frac{1}{1+|at|^\alpha}\left\{|a|^\alpha + (1-|a|^\alpha)\frac{1}{1+|t|^\alpha}\right\}\\ &= \frac{1}{1+|at|^\alpha}\,\frac{1+|a|^\alpha|t|^\alpha}{1+|t|^\alpha}\\ &= \frac{1}{1+|t|^\alpha}.\end{aligned}$$

Hence x_n is distributed as α–Laplace.

If x_0 is arbitrary, then also $\{x_n\}$ is stationary and asymptotically distributed as α–Laplace. This follows since

$$\begin{aligned}\varphi_{x_n}(t) &= \varphi_{x_{n-1}}(at)\left[\frac{1+|a|^\alpha|t|^\alpha}{1+|t|^\alpha}\right]\\ &= \frac{1+|a|^{\alpha n}}{1+|t|^\alpha}\varphi_{x_0}(a^n t)\end{aligned}$$

Hence

$$\varphi_{x_n}(t) \to \frac{1}{1+|t|^\alpha} \quad \text{as} \quad n\to\infty.$$

Remark The model given by (4.6.7) extends the EAR(1) model of Gaver and Lewis (1980) and the MLAR(1) model of Jayakumar and Pillai (1993). When $\alpha = 2$, the variables are distributed as double exponential (Laplace). This model can be used for modelling data relating to stock returns and commodity prices. Rachev and Sen Gupta (1991) describe the use of geometric stable distributions, which is exactly the same as α–Laplace distributions, in modelling stock returns. In the class of geometric stable distributions, Laplace distribution plays the role of normal distribution in the class of stable laws. Anderson and Arnold (1993) also describe the use of the Linnik distribution, which is the same as the α–Laplace distribution, in modelling financial time series.

Exercises 4.6.

4.6.1. Derive the Laplace transform of the exponential tailed distribution.

4.6.2. Derive the innovation structure of the Mittag-Leffler tailed autoregressive process.

4.6.3. Examine whether the Mittag-Leffler tailed distribution is self-decomposable.

4.6.4. Give a real life example where exponential tailed distribution can be used for modelling.

4.6.5. Show that Laplace distribution belongs to the semi-α-Laplace family.

4.6.6. Define a geometric exponential distribution similar to the geometric stable distribution.

4.6.7. Try to develop a generalized Laplacian model, with characteristic function $\varphi_X(t) = \left(\frac{1}{1-\beta^2 t^2}\right)^{\alpha}$.

4.6.8. Develop the concept min geometric infinite divisibility by replacing addition by minimum in the case of g.i.d.

4.6.9. Develop an autoregressive minification structure by replacing addition by minimum in the standard AR(1) equation.

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