CHAPTER 5

RAMANUJAN'S THEORIES OF THETA AND ELLIPTIC FUNCTIONS -II

[This chapter is based on the lectures of Professor S.Bhargava of the Department of Mathematics, University of Mysore, Manasa Gangothri, Mysore -570006, India.]

5.0. Introduction

The present lectures are aimed at covering in detail and abinitio, basics of the theory of cubic theta functions implied in Ramanujan's works and being developed by current day mathematicians including the present author. The lectures are elementary and self-contained and should enable the readers further related reading and research.

5.1. The One-variable Cubic Theta Functions

5.1.1. Series definitions of cubic theta functions and some simple consequences

Definition 5.1.1. One variable cubic theta functions.
We define for $|q| < 1$,

$$\begin{aligned} a(q) &:= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \\ b(q) &:= \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}, \ (\omega = e^{\frac{2\pi i}{3}}), \\ c(q) &:= \sum_{m,n=-\infty}^{\infty} q^{(m+\frac{1}{3})^2+(m+\frac{1}{3})(n+\frac{1}{3})+(n+\frac{1}{3})^2}. \end{aligned}$$

Remark 5.1.1: We note the analogy between the above functions and the one variable classical theta functions (in Ramanujan's and Jacobi's notations, respectively).

$$\begin{aligned} \phi(q) &:= \Theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \\ \phi(-q) &:= \Theta_4(q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}, \\ 2q^{\frac{1}{4}}\psi(q^2) &:= \Theta_2(q) := \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}. \end{aligned}$$

Exercises 5.1.

5.1.1. Discuss the convergence of the series in Definition 5.1.1 and Remark 5.1.1.

5.1.2. Prove

$$\begin{aligned} &\text{(i)} & a(q) &= \phi(q)\phi(q^3) + 4q\psi(q^2)\psi(q^6) \\ &\text{(ii)} & a(q^4) &= \frac{1}{2}[\phi(q)\phi(q^3) + \phi(-q)\phi(-q^3)] \\ &\text{(iii)} & b(q) &= \frac{3}{2}a(q^3) - \frac{1}{2}a(q) \\ &\text{(iv)} & c(q) &= \frac{1}{2}a(q^{\frac{1}{3}}) - \frac{1}{2}a(q) \\ &\text{(v)} & c(q^3) &= \frac{1}{3}[a(q) - b(q)] \\ &\text{(vi)} & b(q) &= a(q^3) - c(q^3). \end{aligned}$$

5.1.2. Product representations for $b(q)$ and $c(q)$

Theorem 5.1.1. *We have*

$$\begin{aligned} &\text{(i)} & b(q) &= \frac{f^3(-q)}{f(-q^3)} = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty} \\ &\text{(ii)} & c(q) &= 3q^{\frac{1}{3}}\frac{f^3(-q^3)}{f(-q)} = 3q^{\frac{1}{3}}\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}, \end{aligned}$$

where, following Ramanujan,

$$f(-q) := \prod_{n=1}^{\infty}(1-q^n) =: (q;q)_\infty.$$

Proof: (J.M. Borwein, P.B. Borwein and F.G. Garvan [7]):
The Euler-Cauchy q-binomial theorem says that, for $|q| < 1$,

$$\frac{(a;q)_\infty}{(b;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a/b)_n}{(q)_n} b^n$$

where, as usual,

$$\begin{aligned}(a)_\infty := (a;q)_\infty : &= \prod_{n=0}^{\infty}(1-aq^n),\\ (a)_n := (a;q)_n : &= \prod_{k=1}^{n}(1-aq^{k-1}).\end{aligned}$$

Letting b to 0 in the q-binomial theorem we have (Euler)

$$(a;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n-1)/2}}{(q)_n}.$$

This gives, since

$$\begin{aligned}(a^3;q^3)_\infty &= (a;q)_\infty (a\omega;q)_\infty (a\omega^2;q)_\infty,\\ \sum_{n=0}^{\infty} \frac{a^{3n} q^{3n(n-1)/2}}{(q^3;q^3)_\infty} &= \sum_{n_1,n_2,n_3=0}^{\infty} \omega^{n_1+2n_2} a^{n_0+n_1+n_3}\\ &\times \frac{q^{[n_0(n_0-1)+n_1(n_1-1)+n_2(n_2-1)]/2}}{(q)_{n_0}(q)_{n_1}(q)_{n_2}}.\end{aligned}$$

Equating the coefficients of like powers of a, we have,

$$\frac{1}{(q^3;q^3)_\infty} = \sum_{n_0+n_1+n_2=3n} \omega^{n_1-n_2} \frac{q^{[n_0(n_0-1)+n_1(n_1-1)+n_2(n_2-1)]/2}}{(q)_{n_0}(q)_{n_1}(q)_{n_2}}.$$

Or, changing n_i to $m_i + n$, $i = 0, 1, 2$,

$$\frac{1}{(q^3;q^3)_\infty} = \sum_{m_0+m_1+m_2=0} \omega^{m_1-m_2} \frac{q^{\frac{1}{2}(m_0^2+m_1^2+m_2^2)}}{(q)_{m_0+n}(q)_{m_1+n}(q)_{m_2+n}}.$$

Letting n to ∞, this gives

$$\sum_{m_1,m_2=-\infty}^{\infty} \omega^{m_1-m_2} q^{m_1^2+m_1m_2+m_2^2} = \frac{(q)_\infty^3}{(q^3;q^3)_\infty}.$$

This indeed is the first of the required results.

We leave the proof of part (ii) as an exercise.

Exercise 5.1.3. Prove the q-binomial theorem.

Exercise 5.1.4. Complete the proof to the second part of Theorem 5.1.1. (See Exercise 5.2.3 for a proof.)

5.1.3. The cubic analogue of Jacobi's quartic modular equations

The following theorem gives two versions of the cubic counterpart of the Jacobi's modular equation

$$\phi^4(q) = \phi^4(-q) + 16q\psi^4(q^2).$$

The first version is Entry (iv) of Chapter 20 in Ramanujan's Second Notebook, and the second is due to J.M. Borwein and P.B. Borwein.

Theorem 5.1.2.

$$3 + \frac{f^3(-q^{\frac{1}{3}})}{q^{\frac{1}{3}} f^3(-q^3)} = \left(27 + \frac{f^{12}(-q)}{q f^{12}(-q^3)}\right)^{\frac{1}{3}},$$

or, what is the same,

$$a^3(q) = b^3(q) + c^3(q).$$

Proof of first version (B.C. Berndt [2].)

We need the following results concerning the classical theta function and its restrictions as found in Chapter 16 of Ramanujan's Second Notebook [1],[11].

$$\begin{aligned}
f(a,b) &:= \sum_{-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \\
&= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \ |ab|<1, \\
\phi(q) &:= f(q,q) = (-q;-q)_\infty(-q;q^2)_\infty^2 = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \\
\chi(q) &:= (-q;q^2)_\infty = [\frac{\phi(q)}{\psi(-q)}]^{\frac{1}{3}} = \frac{\phi(q)}{f(q)}, \\
\psi(q) &:= f(q,q^3) = f(q^3,q^6) + q\psi(q^9).
\end{aligned}$$

These follow by elementary series and product manipulations [1]. Now, setting

$$v = q^{\frac{1}{3}}\chi(-q)/\chi^3(-q^3),$$

We have

$$\begin{aligned}
v^{-1} &= \frac{(-q;q)_\infty\phi(-q^3)}{q^{\frac{1}{3}}\psi(q^3)} \\
&= \frac{(-q^3;q^3)_\infty f(q,q^2)\phi(-q^3)}{q^{\frac{1}{3}}\psi(q^3)(q^3;q^3)_\infty} \\
&= \frac{f(q,q^2)}{q^{\frac{1}{3}}\psi(q^3)} = \frac{\psi(q^{\frac{1}{3}})}{q^{\frac{1}{3}}\psi(q^3)} - 1.
\end{aligned}$$

Thus,

$$1 + v^{-1} = \frac{\psi(q^{\frac{1}{3}})}{q^{\frac{1}{3}}\psi(q^3)}.$$

Similarly,

$$1 - 2v = \frac{\phi(-q^{\frac{1}{3}})}{\phi(-q^3)}.$$

From the last two identities, we have

$$\begin{aligned}
\frac{f^3(-q^{\frac{1}{3}})}{q^{\frac{1}{3}} f^3(-q^3)} &= \frac{\phi^2(-q^{\frac{1}{3}})\psi(q^{\frac{1}{3}})}{q^{\frac{1}{3}}\phi^2(-q^3)\psi(q^3)} \\
&= (1-2v)^2\left(1+\frac{1}{v}\right) = 4v^2 + \frac{1}{v} - 3.
\end{aligned}$$

Changing $q^{\frac{1}{3}}$ to $\omega q^{\frac{1}{3}}$ and $q^{\frac{1}{3}}$ to $\omega^2 q^{\frac{1}{3}}$ in this equation and multiplying it with the resulting two equations we have, on some manipulations,

$$\begin{aligned}\frac{f^{12}(-q)}{q f^{12}(-q^3)} &= \left(4v^2 + \frac{1}{v}\right)^3 - 27\\ &= \left(\frac{f^3(-q^{\frac{1}{3}})}{q^{\frac{1}{3}} f^3(-q^3)} + 3\right)^3 - 27,\end{aligned}$$

which is the desired identity.

Proof of the second version (J.M. Borwein, P.B. Borwein and F.G. Garvan [7].) Firstly, we have, under the transformations $q \to \omega q$,

$$\text{(i)} \ \ a(q^3) = \sum_{m,n=-\infty}^{\infty} q^{3(m^2+mn+n^2)} \to a(q^3)$$

and similarly,

$$\text{(ii)} \ \ b(q^3) \to b(q^3) \text{ and } \ \text{(iii)} \ \ c(q^3) \to \omega c(q^3).$$

Similarly, we have $a(q^3)$, $b(q^3)$ and $c(q^3)$ going respectively to $a(q^3)$, $b(q^3)$ and $\omega^2 c(q^3)$ under $q \to \omega^2 q$. Thus, for these and result (v) of Exercise 5.1.2, we have

$$\begin{aligned}b(\omega q) &= a(q^3) - \omega c(q^3),\\ b(\omega^2 q) &= a(q^3) - \omega^2 c(q^3)\end{aligned}$$

and, therefore,

$$\begin{aligned}b(q)b(\omega q)b(\omega^2 q) &= (a(q^3) - c(q^3))(a(q^3) - \omega c(q^3))\\ &\times (a(q^3) - \omega^2 c(q^3))\\ &= a^3(q^3) - c^3(q^3).\end{aligned}$$

However, on using Part (i) of Theorem 5.1.1, the left side of the last equality equals $b^3(q^3)$ on some manipulations. This completes the proof of the desired identity.

Remark 5.1.2. For still another proof of the theorem within Ramanujan's repertoire, one may see [3].

Remark 5.1.3. (H.H. Chan [8]). That the two versions of the theorem are equivalent follows on employing Theorem 5.1.1 and result (v) of Exercise 5.1.2.

Exercise 5.1.5. Prove the various identities quoted in the proof to Theorem 5.1.2.

Exercise 5.1.6. Complete the manipulations indicated throughout the proof of Theorem 5.1.2.

Exercise 5.1.7. Work out the details under Remark 5.1.3.

5.2. The Two-variable Cubic Theta Functions

5.2.1. Series definitions and some simple consequences

Definition 5.2.1. Two-variable cubic theta functions.

For $|q| < 1, z \neq 0$, we define

$$\begin{aligned}
a(q,z) &:= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} z^{m-n}\\
b(q,z) &= \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2} z^n, \ (\omega = \mathrm{e}^{2\pi i/3}),\\
c(q,z) &:= \sum_{m,n=-\infty}^{\infty} q^{(m+\frac{1}{3})^2+(m+\frac{1}{3})(n+\frac{1}{3})+(n+\frac{1}{3})^2} z^{m-n}\\
a'(q,z) &:= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} z^n.
\end{aligned}$$

Remark 5.2.1 We note that the one-variable functions of Section 5.1 are restrictions of the above functions at $z = 1$. In fact

$$\begin{aligned}
a(q) &= a(q,1) = a'(q,1);\\
b(q) &= b(q,1); \ c(q) = c(q,1).
\end{aligned}$$

Remark 5.2.2. We note the analogy between the above functions and the two-variable classical theta functions (in Ramanujan's and Jacobi's notations, respectively);

$$\begin{aligned}
f(qz, qz^{-1}) &:= \Theta_3(q,z) := \sum_{n=-\infty}^{\infty} q^{n^2} z^n, \\
f(-qz, -qz^{-1}) &:= \Theta_4(q,z) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} z^n, \\
q^{\frac{1}{4}} f(q^2 z, z^{-1}) &:= \Theta_2(q,z) := \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} z^n,
\end{aligned}$$

where, as before, following Ramanujan,

$$f(a,b) := \sum_{-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

Exercises 5.2.

5.2.1 Discuss the convergence of the series in Definition 5.2.1.

5.2.2. Prove,

$$\begin{aligned}
&\text{(i)} & a'(q,z) &= z^2 q^3 a'(q, zq^3), \\
&\text{(ii)} & a(q,z) &= z^2 q a(q, zq), \\
&\text{(iii)} & b(q,z) &= z^2 q^3 b(q, zq), \\
&\text{(iv)} & (q,z) &= z^2 q c(q, zq), \\
&\text{(v)} & a'(q,z) &= a(q^3, z) + 2qc(q^3, z), \\
&\text{(vi)} & b(q,z) &= a(q^3, z) - qc(q^3, z), \\
&\text{(vii)} & a'(q,z) &= b(q,z) + 3qc(q^3, z).
\end{aligned}$$

5.2.2. Product representations for $b(q,z)$ and $c(q,z)$

Theorem 5.2.1. *We have,*

$$\begin{aligned}
\text{(i) } b(q,z) &= (q)_\infty (q^3;q^3)_\infty \frac{(zq)_\infty (z^{-1}q)_\infty}{(zq^3;q^3)_\infty (z^{-1}q^3;q^3)_\infty} \\
\text{(ii) } c(q,z) &= q^{\frac{1}{3}}(1+z+z^{-1})(q)_\infty (q^3;q^3)_\infty \\
&\quad \times \frac{(z^3q^3;q^3)_\infty (z^{-1}q^3;q^3)_\infty}{(zq)_\infty (z^{-1}q)_\infty}.
\end{aligned}$$

Here, $(a)_\infty \;:= (a;q)_\infty$, for brevity.

Proof (Part (i): M. Hirschhorn, F. Garvan and J. Borwein [10].)

We have, by definition of $b(q,z)$,

$$b(q,z) = \sum_{n:even,-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2} z^n + \sum_{n:odd,-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2} z^n.$$

Setting $n = 2k$ in the first sum and $n = 2k+1$ in the second, we get, after slight manipulations,

$$\begin{aligned}
b(q,z) &= \sum_{m=-\infty}^{\infty} \omega^m q^{m^2} \sum_{k=-\infty}^{\infty} q^{3k^2} z^{2k} \\
&\quad + \omega^{-1} qz \sum_{m=-\infty}^{\infty} \omega^m q^{m^2+m} \sum_{k=-\infty}^{\infty} q^{3k^2+3k} z^{2k}.
\end{aligned}$$

Applying Jacobi's triple product identity (the product form of $f(a,b)$ met with in the proof of Theorem 5.1.2) to each of the sums we have, after some recombining of the various products involved,

$$\begin{aligned}
b(q,z) &= \frac{(-q^3;q^6)_\infty (q^2;q^2)_\infty (q^6;q^6)_\infty}{(-q;q^2)_\infty} (-z^2q^3;q^6)_\infty \\
&\quad \times (-z^{-2}q^3;q^6)_\infty \\
&\quad - q(z+z^{-1}) \frac{(-q^6;q^6)_\infty (q^2;q^2)_\infty (q^6;q^6)_\infty}{(-q^2;q^2)_\infty} \\
&\quad \times (-z^2q^6;q^6)_\infty (-z^{-2}q^6;q^6)_\infty.
\end{aligned}$$

On the other hand, one can prove that the right side, say $G(z)$, of the required identity for $b(q,z)$ is precisely the right side of the above identity. One has to first observe that $G(q^3z) = z^{-2}q^{-3}G(z)$ and then use it to show that we can have

$$\begin{aligned} G(z) &= C_0 \sum_{n=-\infty}^{\infty} q^{3n^2} z^{2n} + C_1 z \sum_{n=-\infty}^{\infty} q^{3n^2+3n} z^{2n}, \text{ or,} \\ G(z) &= C_0(-z^2q^3;q^6)_\infty(-z^2q^3;q^6)_\infty(q^6;q^6)_\infty \\ &+ C_1(z+z^{-1})(-z^2q^6;q^6)_\infty(-z^{-2}q^6;q^6)_\infty(q^6;q^6)_\infty. \end{aligned}$$

It is now enough to evaluate C_0 and C_1. For this, one can put successively $z = i$ and $z = iq^{-\frac{3}{2}}$ in the last equation. Now, for Part (ii) of the theorem, we have, by definition of $c(q,z)$,

$$\begin{aligned} c(q,z) &= q^{\frac{1}{3}} \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} z^{m-n} \\ &= q^{\frac{1}{3}} \sum_{n,k=-\infty}^{\infty} q^{(n+2k)^2+(n+2k)n+n^2+2n+2k} z^{2k} \\ &+ q^{\frac{1}{3}} z \sum_{n,k=-\infty}^{\infty} q^{(n+2k+1)^2+(n+2k+1)n+n^2+2n+2k+1} z^{2k+1} \end{aligned}$$

(on separating even and odd powers of z),

$$\begin{aligned} &= q^{\frac{1}{3}} \sum_{n,k=-\infty}^{\infty} q^{3(n+k)^2+2(n+k)+k^2} z^{2k} \\ &+ q^{\frac{1}{3}} z \sum_{n,k=-\infty}^{\infty} q^{3(n+k)^2+5(n+k)+2+k^2+k} z^{2k} \\ &= q^{\frac{1}{3}} \sum_{t,k=-\infty}^{\infty} q^{3t^2+2t} q^k z^{2k} + q^{\frac{1}{3}} z \\ &\times \sum_{t',k=-\infty}^{\infty} q^{(3t'-1)(t'+1)} q^{k^2} (z^2q)^k \end{aligned}$$

(setting $n+k = t$ and $n+k+1 = t'$),

$$= q^{\frac{1}{3}} \sum_{t=-\infty}^{\infty} q^{3t^2+2t} \sum_{k=-\infty}^{\infty} q^{k^2} z^{2k} + q^{\frac{1}{3}} z$$

$$\begin{aligned}&\times \sum_{t=-\infty}^{\infty} q^{3t^2-t} \sum_{k=-\infty}^{\infty} q^{k^2}(z^2q)^k \\ &= q^{\frac{1}{3}} f(q^5,q) f(qz^2,qz^{-2}) + q^{\frac{1}{3}} z f(q^2,q^4) f(q^2z^2,z^{-2}).\end{aligned}$$

Comparing this with the required identity for $c(q,z)$, it is now enough to prove

$$\begin{aligned}G(z) &:= (1+z+z^{-1})\frac{(z^3q^3;q^3)_\infty(z^{-3}q^3;q^3)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty} \\ &= \frac{f(q^5,q)f(qz^2,qz^{-2})}{(q;q)_\infty(q^3;q^3)_\infty} + z\frac{f(q^4,q^2)f(q^2z^2,z^{-2})}{(q;q)_\infty(q^3;q^3)_\infty}. \qquad (5.2.1)\end{aligned}$$

Now, we can write

$$G(z) = \frac{1}{z}\frac{(z^3;q^3)_\infty}{(z;q)_\infty}\frac{(z^{-3}q^3;q^3)_\infty}{(z^{-1}q;q)_\infty}.$$

We have,

$$\begin{aligned}G(qz) &= \frac{1}{qz}\frac{(z^3q^3;q^3)_\infty}{(zq;q)_\infty}\frac{(z^{-3};q^3)_\infty}{(z^{-1};q)_\infty} \\ &= \frac{1}{qz^2}G(z)\end{aligned}$$

so that

$$z^2qG(qz) = G(z).$$

This gives, on seeking

$$G(z) = \sum_{n=-\infty}^{\infty} C_n z^n,$$

$$\sum_{n=-\infty}^{\infty} q^{n+1}C_n z^{n+2} = \sum_{n=-\infty}^{\infty} C_n z^n,$$

and hence

$$C_n = q^{n-1}C_{n-2},\ n = 0, \pm1, \pm2, ...$$

This gives, on iteration,

$$C_{2n} = q^{n^2}C_0,\ C_{2n+1} = q^{n(n+1)}C_1,\ n = 0, \pm1, \pm2, ...$$

Thus, the power series sought for $G(z)$ becomes

$$\begin{aligned}G(z) &= C_0\sum_{n=-\infty}^{\infty} q^{n^2}z^{2n} + C_1 z\sum_{n=-\infty}^{\infty} q^{n^2}(z^2q)^n,\ \text{or,} \\ G(z) &= C_0 f(qz^2,qz^{-2}) + C_1 z f(q^2z^2,z^{-2}). \qquad (5.2.2)\end{aligned}$$

Comparing this with (5.2.1), it is enough to prove,

$$C_0 = \frac{f(q^5,q)}{(q;q)_\infty(q^3;q^3)_\infty}, \text{ and}$$
$$C_1 = \frac{f(q^4,q^2)}{(q;q)_\infty(q^3;q^3)_\infty}.$$

Putting $z = i$ in (5.2.2) and using the definition of $G(z)$ in (5.2.1), we have

$$\begin{aligned} G(i) &= \frac{(-iq^3;q^3)_\infty(iq^3;q^3)_\infty}{(iq;q)_\infty(-iq;q)_\infty} \\ &= C_0 f(-q,q). \end{aligned}$$

or

$$\begin{aligned} C_0 &= \frac{(-q^6;q^6)_\infty}{(-q^2;q^2)_\infty(q;q^2)_\infty(q^2;q^2)_\infty} \\ &= \frac{(q^2;q^4)_\infty(-q^6;q^6)_\infty}{(q;q)_\infty(q;q^2)_\infty} \\ &= \frac{(-q;q^2)_\infty(-q^6;q^6)_\infty}{(q;q)_\infty} \\ &= \frac{(-q;q^6)_\infty(-q^3;q^6)_\infty(-q^5;q^6)_\infty}{(q;q)_\infty} \\ &= \frac{(-q;q^6)_\infty(-q^5;q^6)_\infty(-q^3;q^3)_\infty(-q^6;q^6)_\infty}{(q;q)_\infty} \\ &= \frac{(-q;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty}{(q;q)_\infty(q^3;q^3)_\infty} \\ &= \frac{f(q,q^5)}{(q;q)_\infty(q^3;q^3)_\infty} \end{aligned}$$

as required, as regards C_0. Now, for C_1, putting $z = iq^{-\frac{1}{2}}$ in (5.2.2) and using the defintion of $G(z)$ in (5.2.1), we have.

$$\begin{aligned} G(iq^{-\frac{1}{2}}) &= -iq^{\frac{1}{2}}\frac{(-iq^{-\frac{3}{2}};q^3)_\infty(iq^{\frac{9}{2}};q^3)_\infty}{(iq^{-\frac{1}{2}};q)_\infty(-iq^{\frac{3}{2}};q)_\infty} \\ &= C_1 iq^{-\frac{1}{2}} f(-q,-q). \end{aligned}$$

or,

$$
\begin{aligned}
C_1 &= \frac{-q(-iq^{-\frac{3}{2}};q^3)_\infty(iq^{\frac{9}{2}};q^3)_\infty}{(iq^{-\frac{1}{2}};q)_\infty(-iq^{\frac{3}{2}};q)_\infty(q;q^2)^2_\infty(q^2;q^2)_\infty} \\
&= \frac{-q(1+iq^{\frac{1}{2}})(1+iq^{-\frac{3}{2}})(-iq^{\frac{3}{2}};q^3)_\infty(iq^{\frac{3}{2}};q^3)_\infty}{(1-iq^{-\frac{1}{2}})(1-iq^{\frac{3}{2}})(iq^{\frac{1}{2}};q)_\infty(-iq^{\frac{1}{2}};q)_\infty(q;q)_\infty(q;q^2)_\infty} \\
&= \frac{(-q^3;q^6)_\infty}{(-q;q^2)_\infty(q;q^2)_\infty(q;q)_\infty} = \frac{(-q^3;q^6)_\infty}{(q^2;q^4)_\infty(q;q)_\infty} \\
&= \frac{(-q^3;q^6)_\infty(-q^2;q^2)_\infty}{(q;q)_\infty} \\
&= \frac{(-q^3;q^6)_\infty(-q^6;q^6)_\infty(-q^4;q^6)_\infty(-q^2;q^6)_\infty}{(q;q)_\infty} \\
&= \frac{(-q^3;q^3)_\infty(-q^4;q^6)_\infty(-q^2;q^4)_\infty}{(q;q)_\infty} \\
&= \frac{(q^6;q^6)_\infty(-q^4;q^2)_\infty(-q^2;q^4)_\infty}{(q;q)_\infty(q^3;q^3)_\infty} = \frac{f(q^2,q^4)}{(q;q)_\infty(q^3;q^3)_\infty},
\end{aligned}
$$

as required. This completes the proof of Part (ii) of the theorem and hence that of the theorem.

Exercise 5.2.3 Deduce from Theorem 5.2.1, product representations for $b(q)$ and $c(q)$.

Exercise 5.2.4. (M. Hirschhorn, F. Garvan and J. Borswein [10]).If

$$G(a) := f(azq^{\frac{1}{2}}, a^{-1}z^{-1}q^{\frac{1}{2}})(f(az^{-1}q^{\frac{1}{2}}, a^{-1}zq^{\frac{1}{2}})f(aq^{\frac{1}{2}}, a^{-1}q^{\frac{1}{2}})$$

then show successively that

$$
\begin{aligned}
&\text{(i)} & G(aq) &= a^3q^{-\frac{3}{2}}f(a), \\
&\text{(ii)} & G(a) &= C_0 f(a^3q^{\frac{3}{2}}, a^{-3}q^{\frac{3}{2}}) + C_1[f(a^3q^{\frac{5}{2}}, a^{-3}q^{\frac{1}{2}}) + a^{-1}f(a^3q^{\frac{1}{2}}, a^{-3}q^{\frac{5}{2}})].
\end{aligned}
$$

[Hint: Seek $G(a) = \sum_{n=-\infty}^{\infty} C_n a^n$ and apply Part (i) for a recurrence relation for C_n].

$$\text{(iii)}\quad 3C_0 = \frac{z^{-1}(q)_\infty^3}{(q^3;q^3)_\infty^3} \times \left[\frac{f(-z^{-3}q^3,-z^3)}{f(-z^{-1}q,-z)}\left\{1+6\sum_{n=1}^{\infty}\left(\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right)\right\} - f^2(-z^{-1}q,-z)\right].$$

[Hint: Change a to $a^2q^{\frac{1}{2}}$ in (*ii*), multiply the resulting equation by a^3, apply operator $\Theta_a = a\frac{\mathrm{d}}{\mathrm{d}a}$ after rewriting the equation suitably and let a tend to i in the resulting equation.]

$$\text{(iv)}\quad C_1 = z^{-1}q^{\frac{1}{2}}(q)_\infty^2\frac{f(-z^3,-z^{-3}q^3)}{f(-z,-z^{-1}q)}.$$

[Hint: Change a to $a^2q^{\frac{1}{2}}$ in Part (ii), multiply by a^3 and then let a to $\mathrm{e}^{\frac{\pi i}{6}}$.]

(v) $a(q,z)$ equals constant term in the expansion of $G(a)$ as power series in a. Hence show that

$$\begin{aligned} a(q,z) &= \frac{1}{3}(1+z+z^{-1}) \\ &\times \left\{1+6\sum_{n=1}^{\infty}\left(\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right)\right\} \\ &\times \frac{(q)_\infty^2}{(q^3;q^3)_\infty^2}\frac{(z^3q^3;q^3)_\infty(z^{-3}q^3;q^3)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty} \\ &+ \frac{1}{3}(2-z-z^{-1})\frac{(q)_\infty^5}{(q^3;q^3)_\infty^3}(zq;q)_\infty^2(z^{-1}q;q)_\infty^2. \end{aligned}$$

[Hint: $a(q,z) = \sum_{m+n+p=0,m,n,p=-\infty}^{\infty} q^{(m^2+n^2+p^2)/2}z^{m-n}$.]

(vi) $c(q,z) = q^{\frac{1}{3}}$ constant term in the expansion of $aG(aq^{\frac{1}{2}})$ as power series in a. Hence prove the product representation for $c(q,z)$ given in Theorem 5.2.1.

Exercise 5.2.5. Letting $z = 1$ in Part (v) of Exercise 5.2.4, obtain the "Lambert series" for $a(q)$:

$$a(q) = 1+6\sum_{n=1}^{\infty}\left(\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right).$$

5.2.3. A two-variable cubic counterpart of Jacobi's quartic modular equation

Theorem 5.2.2. *(M.Hirschhorn, F. Garvan and J. Borwein [10].)*
We have

$$a^3(q,z) = b^2(q)b(q,z^3) + qc^3(q,z).$$

Proof: By Part (vi) of Exercise (5.2.2),

$$b(q,z)b(q\omega,z)b(q\omega^2,z) = a^3(q^3,z) - q^3c^3(q^3,z).$$

But, by Part (i) of Theorem 5.2.1, we have the left side of this to be, on slight manipulation, equal to $b^2(q^3)b(q^3,z^3)$. This proves the theorem with q^3 instead of q.

Exercise 5.2.6. (M. Hirschhorn, F. Garvan and J. Borwein [10]).
Show that $\frac{b(q,z)c(q,z)}{b(q^3,z^3)c(q^3,z)}$ is independent of z.

5.3. The Three-variable Cubic Theta Functions

5.3.1. Series definitions of three-variable cubic theta functions and their equivalence, unification of one and two-variable cubic theta functions

Definition 5.3.1. S. Bhargava [4].
If $|q| < 1, \tau, z \neq 0$, we define

$$\begin{aligned}
a(q,\tau,z) &:= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}\tau^{m+n}z^{m-n} \\
b(q,\tau,z) &:= \sum_{m,n=-\infty}^{\infty} \omega^{m-n}q^{m^2+mn+n^2}\tau^m z^n \\
c(q,\tau,z) &:= \sum_{m,n=-\infty}^{\infty} q^{(m+\frac{1}{3})^2+(m+\frac{1}{3})(n+\frac{1}{3})+(n+\frac{1}{3})^2}\tau^{n+m}z^{n-m}
\end{aligned}$$

$$a'(q,\tau,z) \quad := \quad \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}\tau^m z^n.$$

Exercises 5.3.

5.3.1. Show,

$$\begin{aligned} a(q,1,z) &= a(q,z),\ a'(q,1,z) = a'(q,z) \\ b(q,1,z) &= b(q,z),\ c(q,1,z) = c(q,z). \end{aligned}$$

5.3.2. Show,

$$\begin{aligned} a(a) &= a(q,1,1), = a'(q,1,1);\ b(q) = b(q,1,1) \\ c(q) &= c(q,1,1). \end{aligned}$$

Theorem 5.3.1. *(S. Bhargava[4]).*
The four two-variable theta functions are equivalent. In fact,

$$\begin{aligned} a^{'}(q,\tau,z) &= a(q,\ \sqrt{\tau z},\ \sqrt{z/\tau}), \\ b(q,\tau,z) &= a(q,\ \sqrt{\tau z}, \omega^2\ \sqrt{z/\tau}), \\ c(q,\tau,z) &= q^{1/3}a(q,\tau q,z). \end{aligned}$$

Proof: Exercise

Theorem 5.3.2. *(S. Bhargava [4].)*
For any integers λ and μ, we have

$$a(q,\tau,z) = q^{3\lambda^2+3\lambda\mu+\mu^2}\tau^{2\lambda+\mu}\ z^{\mu}\ a(q,\tau q^{3(2\lambda+\mu)/2}, zq^{\mu/2})$$

Proof: Exercise.

Theorem 5.3.3. *(S. Bhargava [4].)*

$$a(q,\tau,z) = a(q^3, \sqrt{\tau^3/z^3}, \sqrt{\tau z^3}) + q\tau z^{-1} a(q^3, q^3\sqrt{\tau^3/z^3}, \sqrt{\tau z^3}).$$

Proof: Exercise.

Exercise 5.3.3. Complete the proof of Theorem 5.3.1. For example,

$$\begin{aligned} a(q, \sqrt{\tau z}, \omega^2\sqrt{z/\tau}) &= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}(\tau z)^{\frac{m+n}{2}} \\ &\times \omega^{2(m-n)}(z/\tau)^{\frac{m-n}{2}} \\ &= \sum_{m,n=-\infty}^{\infty} \omega^{n-m} q^{m^2+mn+n^2}\tau^n z^m \\ &= b(q,\tau,z). \end{aligned}$$

Exercise 5.3.4. Complete the proof of Theorem 5.3.2.
[For a start, expand the right side to get, after some manipulation,

$$\sum_{m,n=-\infty}^{\infty} q^{3(m+\lambda)^2+3(m+\lambda)(n+\mu)+(n+\mu)^2}\tau^{2(m+\lambda)+(n+\mu)}z^{n+\mu}.$$

Then change $m+\lambda$ to m and $n+\mu$ to n.]

Exercise 5.3.5. Complete the proof of Theorem 5.3.3.
[In fact, we can write

$$a(q,\tau,z) = S_0 + S_1 + S_{-1}$$

where

$$S_r = \sum q^{i^2+ij+j^2}\tau^{i+j}z^{j-i}$$

with $i-j = r(mod\ 3)$. Now put $i = j+3m+r$ and manipulate.]

Exercise 5.3.6. Obtain the following identities as special cases of Theorem 5.3.2:

$$\begin{aligned} a'(q,z) &= z^2q^3a'(q,zq^3), \\ a(q,z) &= z^2qa(q,zq), \\ b(q,z) &= z^2q^3b(q,zq^3), \\ c(q,z) &= z^2qc(q,zq). \end{aligned}$$

Exercise 5.3.7. Obtain the counterparts of Theorem 5.3.2 for $a'(q,\tau,z)$,$b(q,\tau,z)$ and $c(q,\tau,z)$.

Exercise 5.3.8. Obtain the following identities as special cases of Theorem 5.3.3.

$$\begin{aligned} a'(q,z) &= a(q^3,z) + 2qc(q^3,z), \\ b(q,z) &= a(q^3,z) - qc(q^3,z). \end{aligned}$$

5.3.2. A representation for $a(q,\tau,z)$ generalizing Hirschhorn - Garvan - Borwein identity

Theorem 5.3.4. *(S. Bhargava [4].)*

$$\begin{aligned} a(q,\tau,z) &= \frac{q^{\frac{1}{2}} f(-q\tau^2, -q^2\tau^{-2}) C_1'(\tau,z)}{6\prod_{n=1}^{\infty}(1-q^{3n}\tau^2)(1-q^{3n}\tau^{-2})(1-q^{3n})S_0(\tau)} \\ &\times \left[1 + 6\sum_{n=1}^{\infty}\left(\frac{q^{3n-2}\tau^2}{1-q^{3n-2}\tau^2} - \frac{q^{3n-1}\tau^{-2}}{1-q^{3n-1}\tau^{-2}}\right)\right] \\ &+ \frac{q^{-\frac{1}{2}}\tau^2 f(-q\tau^{-2}, -q^2\tau^2) C_1'(\tau^{-1},z)}{6\prod_{n=1}^{\infty}(1-q^{3n}\tau^2)(1-q^{3n}\tau^{-2})(1-q^{3n})S_0(\tau)} \\ &\times \left[1 + 6\sum_{n=1}^{\infty}\left(\frac{q^{3n-2}\tau^{-2}}{1-q^{3n-2}\tau^{-2}} - \frac{q^{3n-1}\tau^{2}}{1-q^{3n-1}\tau^{2}}\right)\right] \\ &+ \frac{1}{3}\left[\frac{\tau+\tau^{-1}-z-z^{-1}}{S_0(\tau)}\right]\prod_{n=1}^{\infty}\frac{(1-q^n)^5}{(1-q^{3n})} \\ &\times \prod_{n=1}^{\infty}\frac{(1-q^n\tau z)(1-q^n\tau^{-1}z)(1-q^n\tau z^{-1})}{(1-q^{3n}\tau^2)} \\ &\times \frac{(1-q^n\tau^{-1}z^{-1})}{(1-q^{3n}\tau^{-2})}, \end{aligned}$$

where,

$$C_1'(\tau,z) := \frac{[\omega D(\tau,z) + D(\tau^{-1},z)]\tau^{\frac{2}{3}}q^{\frac{1}{2}}}{\omega+1}$$

with

$$D(\tau,z) := \frac{(-\omega^{-1}\tau^{\frac{1}{3}} + \omega\tau^{-\frac{1}{3}})}{(-\omega^{-1}+\omega)}(z+z^{-1}-\tau^{\frac{1}{3}}\omega^2 - \tau^{-\frac{1}{3}}\omega)$$

$$
\begin{aligned}
&\times \prod_{n=1}^{\infty} \frac{(1-q^{3n})^3}{(1-q^n)} \prod_{n=1}^{\infty} \frac{(1-q^n\omega\tau^{\frac{2}{3}})(1-q^n\omega^2\tau^{-\frac{2}{3}})}{(1+q^n+q^{2n})} \\
&\times \prod_{n=1}^{\infty} \frac{(1-q^n\omega\tau^{-\frac{1}{3}}z)}{(1+q^n+q^{2n})^2} \\
&\times (1-q^n\omega^2\tau^{\frac{1}{3}}z)(1-q^n\omega\tau^{-\frac{1}{3}}z^{-1})(1-q^n\omega^2\tau^{\frac{1}{3}}z^{-1})
\end{aligned}
$$

and

$$\tau^{-1}S_0(\tau) := \frac{\tau+\tau^{-1}}{2} - (\tau-\tau^{-1}) \sum_{n=1}^{\infty} \left(\frac{q^{3n}\tau^2}{1-q^{3n}\tau^2} - \frac{q^{3n}\tau^{-2}}{1-q^{3n}\tau^{-2}} \right).$$

Proof: Proof is similar to that of Part (v) of Exercise 2.4 but more elaborate due to the presence of extra variable τ. We therefore only sketch the proof by indicating the steps involved.

Step (1). Show

$$g(aq,\tau,z) = a^{-3}\tau^{-2}q^{-\frac{3}{2}}g(a,\tau,z),$$

where

$$
\begin{aligned}
g(a,\tau,z) \;&:=\; f(a\tau zq^{\frac{1}{2}}, a^{-1}\tau^{-1}z^{-1}q^{\frac{1}{2}}) f(a\tau z^{-1}q^{\frac{1}{2}}, a^{-1}\tau^{-1}zq^{\frac{1}{2}}) \\
&\times\; f(aq^{\frac{1}{2}}, a^{-1}q^{\frac{1}{2}})
\end{aligned}
$$

Step (2). Show

$$
\begin{aligned}
g(a,\tau,z) \;&=\; C_0(\tau,z) \sum_{n=-\infty}^{\infty} a^{3n}\tau^{2n}q^{3n^2/2} \\
&+\; C_1(\tau,z) \sum_{n=-\infty}^{\infty} a^{3n}\tau^{2n}q^{(3n^2+2n)/2} \\
&+\; C_1(\tau^{-1},z)a^{-1} \sum_{n=-\infty}^{\infty} a^{3n}\tau^{2n}q^{(3n^2-2n)/2}.
\end{aligned}
$$

Step (3). Put each summation in Step (2) in product form.

Step (4). Now, it is easy to see,

$$a(q,\tau,z) = \sum_{m+n+p=0} q^{(m^2+n^2+p^2)/2}\tau^{m+n}z^{m-n}$$

equals the coefficient of a^0 in

$$\left(\sum_{m=-\infty}^{\infty} a^m \tau^m z^m q^{\frac{m^2}{2}}\right)\left(\sum_{n=-\infty}^{\infty} a^n \tau^n z^{-n} q^{\frac{n^2}{2}}\right)\left(\sum_{p=-\infty}^{\infty} a^p q^{\frac{p^2}{2}}\right)$$

equals the coefficient of a^0 in $g(q,\tau,z)$ which is equal to $C_0(\tau,z)$. Now, Similarly,

$$\begin{aligned} a(q,\tau q,z) &= \text{Coefficient of } a^0 \text{ in } ag(aq^{\frac{1}{2}},\tau,z) \\ &= q^{-\frac{1}{2}} C_1(\tau^{-1},z). \end{aligned}$$

Step (5). We have thus proved (after replacing a by $a^2\tau^{\frac{2}{3}}q^{\frac{1}{2}}$ and then multiplying by $a^3\tau^2$),

$$\begin{aligned} a^3\tau^2 f(a^2\tau^{\frac{5}{3}}zq, a^{-2}\tau^{-\frac{5}{3}}z^{-1}) f(a^2\tau^{\frac{5}{3}}z^{-1}q, a^{-2}\tau^{-\frac{5}{3}}z) f(a^2\tau^{\frac{2}{3}}q, a^{-2}\tau^{-\frac{2}{3}}) \\ = a(q,\tau,z)a^3\tau^2 f(a^6\tau^4q^3, a^{-6}\tau^{-4}) \\ + a(q,q\tau,z)\tau^{\frac{4}{3}} a f(a^6\tau^4q^2, a^{-6}\tau^{-4}q) \\ + a(q,q\tau^{-1},z)\tau^{-\frac{4}{3}} a^{-1} f(a^6\tau^4 q, a^{-6}\tau^{-4}q^2). \end{aligned}$$

Step (6). Set $a_1 = -i\omega\tau^{-\frac{2}{3}}$ and $a_2 := i\omega^2\tau^{-\frac{2}{3}}$.
Substitute $a = a_j$, $j = 1,2$ in the identity of Step (5) to obtain two linear simultaneous equations in $a(q,\tau q,z)$ and $a(q,\tau q^{-1},z)$. Eliminating $a(q,\tau q^{-1},z)$, we get $a(q,\tau q,z) = C_1'(\tau,z)$ where $C_1'(\tau,z)$ is as in the statement of the theorem.

Step (7). Now set $a = a_0 := i\tau^{-\frac{1}{3}}$. We have from the identity in Step (5),

$$\begin{aligned} a_0^3\tau^2 f(-\tau z^{-1}q, -\tau^{-1}z) f(-\tau zq, -\tau^{-1}z^{-1}) \lim_{a\to a_0} \frac{f(a^2\tau^{\frac{2}{3}}q, a^{-2}\tau^{-\frac{2}{3}})}{a-a_0} \\ = 3a_0^2\tau^2 a(q,\tau,z) f(-\tau^{-2}, \tau^{-2}q^3) \\ \times\left[1 + \frac{1}{3}a_0\frac{\mathrm{d}}{\mathrm{d}a_0}\log f(a^{-6}\tau^{-4}, a^6\tau^4q^3)\right] \\ + a(q,\tau q,z)\tau^{4/3} f(-\tau^2q^2, -\tau^{-2}q) \\ \times\left[1 + a_0\frac{\mathrm{d}}{\mathrm{d}a_0}\log f(a^6\tau^4q^2, a^{-6}\tau^{-4}q)\right] \end{aligned}$$

$$
\begin{aligned}
- \quad & a(q,\tau^{-1}q,z)a_0^{-2}\tau^{-4/3} f(-\tau^{-2}q^2,-\tau^2 q) \qquad (5.3.1)\\
\times \quad & \left[1 - a_0 \frac{\mathrm{d}}{\mathrm{d}a_0} \log f(a^6\tau^4 q, a^{-6}\tau^{-4}q^2)\right].
\end{aligned}
$$

This yields the expression for $a(q,\tau,z)$ stated in the theorem.

Exercise 5.3.9. Letting $\tau \to 1$ in Theorem 5.3.4, deduce Hirschhorn -Garvan -Borwein [10] representations for $a(q,\tau)$ and $c(q,z)$.

Exercise 5.3.10. Complete the proof of each step in Theorem 5.3.4 following the directions therein.

5.3.3. Two -parameter cubic theta functions in terms of classical theta functions (or, Laurent's expansions for two-parameter cubic theta functions)

Theorem 5.3.5. *(S. Bhargava and S.N. Fathima [5].)*

$$
\begin{aligned}
\text{(i)} \quad a(q,\tau,z) &= f(q^3\tau^2, q^3\tau^{-2}) f(qz^2, qz^{-2})\\
&\quad + q\tau z f(q^6\tau^2, \tau^{-2}) f(q^2z^2, z^{-2}),\\
\text{(ii)} \quad b(q,\tau,z) &= f(q\tau\omega, q\tau^{-1}\omega^2) f(q^3\tau^{-1}z^2, q^3\tau z^{-2})\\
&\quad + \omega^2 qz f(q^3\tau\omega, \tau^{-1}\omega^2) f(q^6\tau^{-1}z^2, \tau z^{-2}),\\
\text{(iii)} \quad c(q,\tau,z) &= q^{\frac{1}{3}} f(q^5\tau^2, q\tau^{-2}) f(qz^2, qz^{-2})\\
&\quad + q^{\frac{7}{3}} \tau z f(q^8\tau^2, q^{-2}\tau^{-2}) f(q^2z^2, z^{-2})\\
\text{(iv)} \quad a'(q,\tau,z) &= f(q\tau, q\tau^{-1}) f(q^3\tau^{-1}z^2, q^3\tau z^{-2})\\
&\quad + qz f(q^2\tau, \tau^{-1}) f(q^6\tau^{-1}z^2, \tau z^{-2}).
\end{aligned}
$$

Proof: We have,

$$
\text{(i)} \quad a(q,\tau,z) = \sum_{m,n=-\infty}^{\infty} q^{n^2+nm+m^2} \tau^{n+m} z^{n-m}
$$

equals the sum of terms with even powers z^{2k}+ sum of terms with odd powers z^{2k+1}, which is

$$
\begin{aligned}
&= \sum_{m,k=\infty}^{\infty} q^{3(m+k)^2+k^2}\tau^{2(m+k)}z^{2k} + q\tau z \sum_{m,k=-\infty}^{\infty} q^{3(m+k)^2+3(m+k)+k^2+k} \\
&= \sum_{n=-\infty}^{\infty} q^{3n^2}\tau^{2n} \sum_{k=-\infty}^{\infty} q^{k^2}z^{2k} + q\tau z \sum_{n=-\infty}^{\infty} q^{3n^2+3n}\tau^{2n} \sum_{k=-\infty}^{\infty} q^{k^2+k}z^{2k} \\
&= f(q^3\tau^2, q^3\tau^{-2})f(qz^2, qz^{-2}) + q\tau z f(q^6\tau^2, \tau^{-2})f(q^2z^2, z^{-2}).
\end{aligned}
$$

$$
\text{(ii)}\ \ b(q,\tau,z) = \sum_{m,n=-\infty}^{\infty} \omega^{m-n}q^{m^2+mn+n^2}\tau^m z^n
$$

equals the part with even powers of z+ part with odd powers of z, which is

$$
\begin{aligned}
&= \sum_{k,m=-\infty}^{\infty} \omega^{k+m}\tau^{3k^2+(k+m)^2}\tau^{k+m}(z^2\tau^{-1})^k \\
&+ qz\omega^2 \sum_{k,m=-\infty}^{\infty} \omega^{m+k}q^{3k^2+3k+(m+k)^2}(q\tau)^{m+k}(z^2\tau^{-1})^k \\
&= \sum_{n,k=-\infty}^{\infty} \omega^n q^{3k^2+n^2}\tau^n(z^2\tau^{-1})^k \\
&+ qz\omega^2 \sum_{n,k=-\infty}^{\infty} q^{n^2}(\tau\omega q)^n q^{3k^2}(q^3z^2\tau^{-1})^k \\
&= f(q\omega\tau, q\omega^2\tau^{-1})f(q^3\tau^{-1}z^2, q^3\tau z^{-2}) \\
&+ qz\omega^2 f(q^2\tau\omega, \tau^{-1}\omega^2)f(q^6z^2\tau^{-1}, z^{-2}\tau), \text{ as desired.}
\end{aligned}
$$

(iii) We have $c(q,\tau,z) = q^{\frac{1}{3}}a(q,q\tau,z)$. Using this in Part (i) we have the required result.

(iv) We have,

$$
a'(q,\tau,z) = b(q,\tau\omega^2,z\omega).
$$

Using this in Part (ii), we have the required result.

Exercise 5.3.11. Putting $\tau = 1$ in Theorem 5.3.5, obtain the corresponding results [S. Cooper [9]) for $a(q,z), b(q,z), c(q,z)$ and $a^{'}(q,z)$.

Exercise 5.3.12. Combining Part (i) of Theorem 5.3.5 with Theorem 5.3.1, get alternative representation for $a^{'}(q,\tau,z), b(q,\tau,z)$ and $c(q,\tau,z)$.

Exercise 5.3.13. (S. Bhargava [4].) Show that

$$\begin{aligned} a(q,\tau,z) &= f(q\tau z^{-1}, q\tau^{-1}z)f(q^3\tau z^3, q^3\tau^{-1}z^{-3}) \\ &+ q\tau z f(q^2\tau z^{-1}, \tau^{-1}z)f(q^6\tau z^3, \tau^{-1}z^{-3}). \end{aligned}$$

[Hint : Write $a(q,\tau,z) = S_0 + S_1$, $j = 2m + r$, where

$$S_r = \sum q^{i^2+\frac{1}{4}j^2+\frac{3}{4}j^2}(\tau/z)^{i+\frac{1}{2}j}(\tau z^3)^{\frac{j}{2}} \;]$$

Exercise 5.3.14. Write the counterparts of Exercise 5.3.13 for $b(q,\tau,z), c(q,\tau,z)$ and $a^{'}(q,\tau,z)$.

Theorem 5.3.6. *(S. Bhargava and S.N. Fathima [6].)*

$$a(e^{-2\pi t}, e^{i\phi}, e^{i\theta}) = \frac{1}{t\sqrt{3}}\exp\left[-\left(\frac{\phi^2+3\theta^2}{6\pi t}\right)\right]a(e^{-\frac{2\pi}{3t}}, e^{\frac{\theta}{t}}, e^{\frac{\phi}{3t}}).$$

Proof: If, as before,

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}.$$

We need the following transform [Entry 20, Chapter 16, [1], [11]],

$$\sqrt{\alpha}f(e^{-\alpha^2+n\alpha}, e^{-\alpha^2-n\alpha}) = e^{n^2/4}\sqrt{\beta}f(e^{-\beta^2+in\beta}, e^{-\beta^2-in\beta})$$

provided $\alpha\beta = \pi$ and Re $(\alpha^2) > 0$. In particular, we need

$$f(e^{-\pi t+i\theta}, e^{-\pi t-i\theta}) = \frac{1}{\sqrt{t}}\exp(-\frac{\theta^2}{4\pi t})f(e^{-\frac{\pi+\theta}{t}}, e^{-\frac{\pi-\theta}{t}}),$$

and

$$f(e^{-\pi t+i\theta}, e^{-i\theta}) = \sqrt{\frac{2}{t}}\exp(\frac{\pi t}{8} - \frac{i\theta}{2} - \frac{\theta^2}{2\pi t})f(-e^{-\frac{2\pi+2\theta}{t}}, -e^{-\frac{2\pi-2\theta}{t}}).$$

We also need the addition results (Entries 30(ii) and 30(iii), Chapter 16, [1], [11])

$$f(a,b) + f(-a,-b) = 2f(a^3b, ab^3)$$

and

$$f(a,b) - f(-a,-b) = 2af(\frac{b}{a}, \frac{a}{b}a^4b^4).$$

We have from Exercise 5.3.13 and repeated use of the above transforms for $f(a,b)$,

$$\begin{aligned} a(e^{-2\pi t}, e^{i\phi}, e^{i\theta}) &= f(e^{-2\pi t + i(\phi-\theta)}, e^{-2\pi t - i(\phi-\theta)}) \\ &\times f(e^{-6\pi t + i(\phi+3\theta)}, e^{-6\pi t - i(\phi+3\theta)}) \\ &+ e^{-2\pi t + i(\phi+\theta)} f(e^{-4\pi t + i(\phi-\theta)}, e^{-i(\phi-\theta)}) \\ &\times f(e^{-12\pi t + i(\phi+3\theta)}, e^{-i(\phi+3\theta)}) \\ &= \frac{1}{2\sqrt{3}t} \exp\left[-\left(\frac{\phi^2+3\theta^2}{6\pi t}\right)\right](\alpha\beta + \alpha'\beta') \end{aligned}$$

where

$$\begin{aligned} \alpha &= f(e^{-\frac{\pi+\phi-\theta}{2t}}, e^{-\frac{\pi-\phi+\theta}{2t}}), \\ \beta &= f(e^{-\frac{\pi+\phi+3\theta}{6t}}, e^{-\frac{\pi-\phi-3\theta}{6t}}), \\ \alpha' &= f(-e^{-\frac{\pi+\phi-\theta}{2t}}, -e^{-\frac{\pi-\phi+\theta}{2t}}), \\ \beta' &= f(-e^{-\frac{\pi+\phi+3\theta}{6t}}, -e^{\frac{\pi-\phi-3\theta}{6t}}). \end{aligned}$$

This becomes, on using the addition theorems for f(a,b) quoted above and the trivial identity

$$\begin{aligned} 2(\alpha\beta + \alpha'\beta') &= (\alpha+\alpha')(\beta+\beta') + (\alpha-\alpha')(\beta-\beta'), \\ a(e^{-2\pi t}, e^{i\phi}, e^{i\theta}) &= \frac{1}{t\sqrt{3}} \exp\left[-\left(\frac{\phi^2+3\theta^2}{6\pi t}\right)\right] \\ &\times \left[f\left(e^{-\frac{2\pi-\theta+\phi}{t}}, e^{-\frac{2\pi+\theta-\phi}{t}}\right) f\left(e^{-\frac{2\pi+3\theta+\phi}{3t}}, e^{-\frac{2\pi-3\theta-\phi}{3t}}\right)\right. \\ &+ \left. e^{-\frac{2\pi+2\theta}{3t}} f\left(e^{\frac{\phi-\theta}{t}}, e^{-\frac{4\pi-\theta+\phi}{t}}\right) f\left(e^{\frac{\phi+3\theta}{3t}}, e^{-\frac{4\pi+\phi+3\theta}{3t}}\right)\right] \\ &= \frac{1}{t\sqrt{3}} \exp\left[-\left(\frac{\phi^2+3\theta^2}{6\pi t}\right)\right] \\ &\times [\ f(q^3\tau z^3, q^3\tau^{-1}z^{-3}) f(q\tau z^{-1}, q\tau^{-1}z) \\ &+ q\tau z f(q^6\tau z^3, \tau^{-1}z^{-3}) f(q^2\tau z^{-1}, \tau^{-1}z)\ ] \end{aligned}$$

with $q = e^{-\frac{2\pi}{3t}}, \tau = e^{-\frac{\theta+\phi}{2t}}, z = e^{\frac{3\theta-\phi}{6t}}$. This reduces to the required identity on using Exercise 5.3.13 once again, the trivial identity $a(q,\tau,z) = a(q,\tau^{-1},z^{-1})$ and the easily verified identity

$$a(q, x^3y, xy^{-1}) = a(q, y^2, x^2).$$

Exercise 5.3.15 Work out all the details in the proof of Theorem 5.3.6.

Exercise 5.3.16. Prove the mixed transformations

$$\text{(i)} \quad a(\mathrm{e}^{-2\pi t}, \mathrm{e}^{i\phi}, \mathrm{e}^{i\theta}) = \frac{1}{t\sqrt{3}} \exp\left[-\left[\frac{\phi^2+3\theta^2}{6\pi t}\right]\right] \times a'\left(\mathrm{e}^{-\frac{2\pi}{3t}}, \mathrm{e}^{\frac{2\phi}{3t}}, \mathrm{e}^{\frac{3\theta+\phi}{3t}}\right),$$

$$\text{(ii)} \quad a'(\mathrm{e}^{-2\pi t}, \mathrm{e}^{i\phi}, \mathrm{e}^{i\theta}) = \frac{1}{t\sqrt{3}} \exp\left[-\left(\frac{\phi^2-\phi\theta+3\theta^2}{6\pi t}\right)\right] \times a\left(\mathrm{e}^{-\frac{2\pi}{3t}}, \mathrm{e}^{\frac{\phi}{2t}}, \mathrm{e}^{\frac{2\theta-\phi}{6t}}\right),$$

$$\text{(iii)} \quad b(\mathrm{e}^{-2\pi t}, \mathrm{e}^{i\phi}, \mathrm{e}^{i\theta}) = \frac{1}{t\sqrt{3}} \exp\left[-\left(\frac{\phi^2-\phi\theta+\theta^2}{6\pi t}\right) + \frac{\phi}{3t}\right] \times c\left(\mathrm{e}^{-\frac{2\pi}{3t}}, \mathrm{e}^{\frac{\phi}{2t}}, \mathrm{e}^{\frac{2\theta-\phi}{6t}}\right),$$

$$\text{(iv)} \quad \exp(-\frac{2i\phi}{3})c(\mathrm{e}^{-2\pi t}, \mathrm{e}^{i\phi}, \mathrm{e}^{i\theta}) = \frac{1}{t\sqrt{3}} \exp\left[-(\frac{\phi^2+3\theta^2}{6\pi t})\right] \times b\left(\mathrm{e}^{-\frac{2\pi}{3t}}, \mathrm{e}^{\frac{2\phi}{3t}}, \mathrm{e}^{\frac{3\theta+\phi}{3t}}\right).$$

[Hint: Use Theorem 5.3.1 on Theorem 5.3.6].

Exercise 5.3.17. Obtain S. Cooper's [9] modular transformations by putting $\phi = 0$ in Theorem 5.3.6 and Exercise 5.3.16.

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