CHAPTER 7

SPECIAL FUNCTIONS OF MATRIX ARGUMENT

[*This Chapter is based on the lectures of Professor A.M. Mathai of McGill University, Canada (Director of the 3rd SERC School).*]

7.0. Introduction

Real scalar functions of matrix argument, when the matrices are real, will be dealt with. It is difficult to develop a theory of functions of matrix argument for general matrices. Let $X = (x_{ij}), i = 1 \cdots, m$ and $j = 1, \cdots, n$ be an $m \times n$ matrix where the x_{ij}'s are real elements. It is assumed that the readers have the basic knowledge of matrices and determinants. The following standard notations will be used here. A prime denotes the transpose, $X' =$ transpose of X, $|(.)|$ denotes the determinant of (.). The same notation will be used for the absolute value also. $\mathrm{tr}(X)$ denotes the trace of X, $\mathrm{tr}(X) =$ sum of the eigenvalues of X. A real symmetric positive definite X will be denoted by $X = X' > 0$. Then $0 < X = X' < I \Rightarrow X = X' > 0$ and $I - X > 0$. Further, $\mathrm{d}X$ will denote the wedge product or skew symmetric product of the differentials $\mathrm{d}x_{ij}$'s.

7.1. Wedge Product and Jacobians

Definition 7.1.1. Wedge product or skew symmetric product of differentials.

Let x and y be real scalar variables with $\mathrm{d}x, \mathrm{d}y$ denoting the differentials. Then

$$\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x \qquad (7.1.1)$$

where $\wedge$ denotes the wedge product or skew symmetric product.

From the definition itself it is clear that $\mathrm{d}x \wedge \mathrm{d}x = 0$. As a consequence, we have the following interesting result: Let $y_1 = f_1(x_1, x_2)$ and $y_2 = f_2(x_1, x_2)$ be two functions of the real scalar variables x_1 and x_2. Then the differentials in y_1 and y_2 are given by the standard formulae, where for example, $\frac{\partial f}{\partial x}$ denotes the partial derivative of f with respect to x,

$$\mathrm{d}y_1 = \frac{\partial f_1}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_1}{\partial x_2}\mathrm{d}x_2$$

and

$$\mathrm{d}y_2 = \frac{\partial f_2}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_2}{\partial x_2}\mathrm{d}x_2. \tag{7.1.2}$$

Let us take the wedge product of $\mathrm{d}y_1$ and $\mathrm{d}y_2$.

$$\begin{aligned}\mathrm{d}y_1 \wedge \mathrm{d}y_2 &= \left[\frac{\partial f_1}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_1}{\partial x_2}\mathrm{d}x_2\right] \wedge \left[\frac{\partial f_2}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_2}{\partial x_2}\mathrm{d}x_2\right]\\ &= \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_1}\mathrm{d}x_1 \wedge \mathrm{d}x_1 + \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}\mathrm{d}x_1 \wedge \mathrm{d}x_2\\ &+ \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}\mathrm{d}x_2 \wedge \mathrm{d}x_1 + \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_2}\mathrm{d}x_2 \wedge \mathrm{d}x_2\\ &= \left[\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}\right]\mathrm{d}x_1 \wedge \mathrm{d}x_2 \end{aligned} \tag{7.1.3}$$

since $\mathrm{d}x_1 \wedge \mathrm{d}x_1 = 0,\ \mathrm{d}x_2 \wedge \mathrm{d}x_2 = 0,\ \mathrm{d}x_2 \wedge \mathrm{d}x_1 = -\mathrm{d}x_1 \wedge \mathrm{d}x_2$. But

$$\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix} = \left|\left(\frac{\partial f_i}{\partial x_j}\right)\right| \tag{7.1.4}$$

$$= \text{ determinant of the Jacobian matrix } \left(\frac{\partial f_i}{\partial x_j}\right)$$

and this determinant is called the *Jacobian* of the transformation of (x_1, x_2) going to (y_1, y_2). Then

$$\mathrm{d}y_1 \wedge \mathrm{d}y_2 = J\ \mathrm{d}x_1 \wedge \mathrm{d}x_2,\ J = \left|\left(\frac{\partial y_i}{\partial x_j}\right)\right|.$$

This property holds in general. Let $y_j = f_j(x_1, ..., x_k),\ j = 1, ..., k$ be k scalar functions of the real scalar variables $x_1, ..., x_k$. Consider the matrix of partial derivatives

$(\frac{\partial y_i}{\partial x_j})$, $i = 1, ..., k$, $j = 1, ..., k$, that is, the (i, j)th element or the i-th row, j-th column element in this matrix is the partial derivative of y_i with respect to x_j. The determinant of this matrix is the Jacobian $J = |(\frac{\partial y_i}{\partial x_j})|$. Then

$$\mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge ... \wedge \mathrm{d}y_k = J\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge ... \wedge \mathrm{d}x_k. \tag{7.1.5}$$

If the transformation $(x_1, ..., x_k)$ to $(y_1, ..., y_k)$ is one-to-one then $|J| \neq 0$. In this case

$$\mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_k = \frac{1}{J}\mathrm{d}y_1 \wedge ... \wedge \mathrm{d}y_k. \tag{7.1.6}$$

If $X = (x_{ij})$ and $m \times n$ then

$$\mathrm{d}X = \mathrm{d}x_{11} \wedge ... \wedge \mathrm{d}x_{1n} \wedge \mathrm{d}x_{21} \wedge ... \wedge \mathrm{d}x_{2n} \wedge ... \wedge \mathrm{d}x_{m1} \wedge ... \wedge \mathrm{d}x_{mn}.$$

If $X = X'$, that is symmetric and $p \times p$, then

$$\mathrm{d}X = \mathrm{d}x_{11} \wedge \mathrm{d}x_{21} \wedge \mathrm{d}x_{22} \wedge ... \wedge \mathrm{d}x_{p1} \wedge ... \wedge \mathrm{d}x_{pp}$$

the wedge product of the $p(p+1)/2$ differentials in X.

Example 7.1.1. Let X and Y be $p \times 1$ vectors of real scalar variables, functionally independent, and let $Y = AX$, $|A| \neq 0$, where $A = (a_{ij})$ is a matrix of constants (free of the elements in X and Y). Evaluate the Jacobian of this transformation.

Solution:

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} = AX = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} \Rightarrow$$

$$y_i = a_{i1}x_1 + ... + a_{ip}x_p, i = 1, ..., p.$$

$$\frac{\partial y_i}{\partial x_j} = a_{ij} \Rightarrow \left(\frac{\partial y_i}{\partial x_j}\right) = (a_{ij}) = A \Rightarrow J = |A|.$$

Hence,

$$\mathrm{d}Y = |A|\mathrm{d}X.$$

That is, Y and X, $p \times 1$, A is $p \times p$, $|A| \neq 0$, A is a constant matrix, then

$$Y = AX,\ |A| \neq 0 \Rightarrow \mathrm{d}Y = |A|\mathrm{d}X. \tag{7.1.7}$$

Example 7.1.2. Let X and Y be $m \times n$ and let $A, m \times m$ and $B, n \times n,$ be nonsingular constant matrices. Then show that

$$Y = AXB \Rightarrow \mathrm{d}Y = |A|^n |B|^m \mathrm{d}X. \tag{7.1.8}$$

Solution: Let $Z = AX = (AX^{(1)}, AX^{(2)}, ..., AX^{(n)})$ where $X^{(1)}, ..., X^{(n)}$ are the columns of X. Then the Jacobian matrix for X going to Y is of the form

$$\begin{bmatrix} A & O & ... & O \\ O & A & ... & O \\ \vdots & \vdots & ... & \vdots \\ O & O & ... & A \end{bmatrix} \Rightarrow \begin{vmatrix} A & O & ... & O \\ \vdots & \vdots & ... & \vdots \\ O & O & ... & A \end{vmatrix} = |A|^n = J_1 \tag{7.1.9}$$

where O denotes a null matrix and J_1 is the Jacobian for the transformation of X going to Z or $\mathrm{d}Z = |A|^n \mathrm{d}X$. Now, consider the transformation

$$Y = ZB = \begin{bmatrix} Z^{(1)}B \\ \vdots \\ Z^{(m)}B \end{bmatrix}$$

where $Z^{(1)}, ..., Z^{(m)}$ are the rows of Z. The Jacobian matrix is of the form,

$$\begin{bmatrix} B & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & B \end{bmatrix} \Rightarrow \begin{vmatrix} B & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & B \end{vmatrix} = |B|^m \Rightarrow \mathrm{d}Y = |B|^m \mathrm{d}Z.$$

Then

$$Y = AXB, |A| \neq 0, |B| \neq 0 \text{ , } Y, m \times n, X, m \times n, \Rightarrow \mathrm{d}Y = |A|^n |B|^m \mathrm{d}X. \tag{7.1.10}$$

Example 7.1.3. Let X be $p \times p$, symmetric positive definite and let $T = (t_{ij})$ be a lower triangular matrix. Consider the transformation

$$X = TT'.$$

Obtain the conditions for this transformation to be one-to-one and then evaluate the Jacobian.

Solution:

$$X = (x_{ij}) = \begin{bmatrix} x_{11} & x_{12} & ... & x_{1p} \\ \vdots & \vdots & ... & \vdots \\ x_{p1} & x_{p2} & ... & x_{pp} \end{bmatrix}$$

with $x_{ij} = x_{ji}$ for all i and j, $X = X' > 0$. When X is positive definite, that is, $X > 0$ then $x_{jj} > 0$, $j = 1, ..., p$ also.

$$TT' = \begin{bmatrix} t_{11} & 0 & ... & 0 \\ t_{21} & t_{22} & ... & 0 \\ \vdots & \vdots & ... & \vdots \\ t_{p1} & t_{p2} & ... & t_{pp} \end{bmatrix} \begin{bmatrix} t_{11} & t_{21} & ... & t_{p1} \\ 0 & t_{22} & ... & t_{p2} \\ \vdots & \vdots & ... & \vdots \\ 0 & 0 & \cdots & t_{pp} \end{bmatrix} = X \Rightarrow$$

$x_{11} = t_{11}^2 \Rightarrow t_{11} = \pm\sqrt{x_{11}}$. This can be made unique if we impose the condition $t_{11} > 0$. Note that $x_{12} = t_{11}t_{21}$ and this means that t_{21} is unique if $t_{11} > 0$. Continuing like this, we see that for the transformation to be unique it is sufficient that $t_{jj} > 0$, $j = 1, ..., p$. Now, observe that,

$$x_{11} = t_{11}^2, x_{22} = t_{21}^2 + t_{22}^2, ..., x_{pp} = t_{p1}^2 + ... + t_{pp}^2$$

and $x_{12} = t_{11}t_{21}, ..., x_{1p} = t_{11}t_{p1}$, and so on.

$$\begin{aligned} \frac{\partial x_{11}}{\partial t_{11}} &= 2t_{11}, \frac{\partial x_{11}}{\partial t_{21}} = 0, ..., \frac{\partial x_{11}}{\partial t_{p1}} = 0, \\ \frac{\partial x_{12}}{\partial t_{21}} &= t_{11}, ..., \frac{\partial x_{1p}}{\partial t_{p1}} = t_{11}, \\ \frac{\partial x_{22}}{\partial t_{22}} &= 2t_{22}, \frac{\partial x_{22}}{\partial t_{31}} = 0, \cdots, \frac{\partial x_{22}}{\partial t_{p1}} = 0, \end{aligned}$$

and so on. Taking the x_{ij}'s in the order $x_{11}, x_{12}, \cdots, x_{1p}, x_{22}, \cdots, x_{2p}, \cdots, x_{pp}$ and the t_{ij}'s in the order $t_{11}, t_{21}, t_{22}, \cdots, t_{pp}$ we have the Jacobian matrix a triangular matrix with the diagonal elements as follows: t_{11} is repeated p times, t_{22} is repeated $p-1$ times and so on, and finally t_{pp} appearing once. The number 2 is appearing a total of p times. Hence the determinant is the product of the diagonal elements, giving,

$$2^p t_{11}^p t_{22}^{p-1} \cdots t_{pp}.$$

Therefore, for $X = X' > 0, T = (t_{ij}),\ t_{ij} = 0, i < j,\ t_{jj} > 0, j = 1, \cdots p$ we have

$$X = TT' \Rightarrow \mathrm{d}X = 2^p \left\{ \prod_{j=1}^{p} t_{jj}^{p+1-j} \right\} \mathrm{d}T. \tag{7.1.11}$$

The transformation in (7.1.11) is a nonlinear transformation whereas in (7.1.10) it is a general linear transformation involving *mn* functionally independent real x_{ij}'s. If X is $p \times p$ and symmetric then there are only $1 + 2 + \cdots + p = p(p+1)/2$ functionally independent elements in X because, here $x_{ij} = x_{ji}$ for all i and j. Let $Y = Y' = AXA', X = X', |A| \neq 0$. Then we can obtain the following result:

$$Y = AXA', X = X', |A| \neq 0, \Rightarrow \mathrm{d}Y = |A|^{p+1} \mathrm{d}X. \tag{7.1.12}$$

This result can be proved by using the fact that a nonsingular matrix such as A can be written as a product of elementary matrices in the form

$$A = E_1 E_2 \cdots E_k$$

where $E_1, \cdots, E_k$ are elementary matrices. Then

$$Y = AXA' \Rightarrow E_1 E_2 \cdots E_k X E_k' \cdots E_1'.$$

Let $Y_k = E_k X E_k', Y_{k-1} = E_{k-1} Y_k E_{k-1}'$, and so on. Evaluate the Jacobians in these transformations to obtain the result in (7.1.12).

For various types of matrix transformations and the associated Jacobians see Mathai (1997). We will need a few more Jacobians for our discussion. These are listed below without proofs.

$$\begin{aligned} Y = X^{-1}, |X| \neq 0 \Rightarrow \mathrm{d}Y &= |X|^{-2p} \text{ for a general } X \\ &= |X|^{-(p+1)} \text{ for } X = X'. \end{aligned} \tag{7.1.13}$$

Let X be a $p \times n, n \geq p$, matrix of rank p and let $X = TU_1'$ where T is $p \times p$ lower triangular with distinct nonzero diagonal elements and U_1' a unique $n \times p$ semiorthonormal matrix, $U_1' U_1 = I_p$, all are of functionally independent variables. Then

$$X = TU_1' \Rightarrow \mathrm{d}X = \left\{ \prod_{j=1}^{p} |t_{jj}|^{n-j} \right\} \mathrm{d}T \wedge \mathrm{d}U_1 \tag{7.1.14}$$

where

$$\int dU_1 = \frac{2^p \pi^{\frac{pn}{2}}}{\Gamma_p\left(\frac{n}{2}\right)}. \tag{7.1.15}$$

(see page 236 for $\Gamma_p(\cdot)$). Combining (7.1.15), (7.1.14) and (7.1.11) we have the following result: Let X be a $p \times n$, $n \geq p$ matrix of full rank p. Let $S = XX' > 0$. Then

$$dX = \frac{\pi^{\frac{np}{2}}}{\Gamma_p\left(\frac{n}{2}\right)} |S|^{\frac{n}{2} - \frac{p+1}{2}} dS. \tag{7.1.16}$$

Exercises 7.1.

7.1.1. Let $X = (x_{ij})$ be $m \times n$ with x_{ij}'s functionally independent. Let A be an $m \times m$ nonsingular constant matrix. Then show that

$$Y = AX \Rightarrow dY = |A|^n dX.$$

7.1.2. Let X be as defined in Exercise 7.1.1. Let B be a $n \times n$ nonsingular constant matrix. Then show that

$$Y = XB \Rightarrow dY = |B|^m dX.$$

7.1.3. Let $X' = -X$ a $p \times p$ skew symmetric matrix of functionally independent real variables. Let A be a nonsingular constant matrix. Then show that

$$Y = AXA' \Rightarrow dY = |A|^{p-1} dX. \tag{7.1.17}$$

7.1.4. Let X be a $p \times p$ skew symmetric nonsingular matrix. Then show that

$$Y = X^{-1} \Rightarrow dY = |X|^{-(p-1)} dX. \tag{7.1.18}$$

7.1.5 Let $X = X' > 0$ be $p \times p$. Let $T = (t_{ij})$ be an upper triangular matrix with positive diagonal elements. Then show that

$$X = TT' \Rightarrow dX = 2^p \left\{ \prod_{j=1}^{p} t_{jj}^{j} \right\} dT. \tag{7.1.19}$$

7.1.6. Let $x_1, \cdots, x_p$ be real scalar variables. Let $y_1 = x_1 + \cdots + x_p$, $y_2 = x_1x_2 + x_1x_3 + \cdots x_{p-1}x_p$ (sum of products taken two at a time), $\cdots, y_k = x_1 \cdots x_k$. Then for $x_j > 0, j = 1, \cdots, k$ show that

$$\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_k = \left\{\prod_{i=1}^{p-1} \prod_{j=i+1}^{p} |x_i - x_j|\right\} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p. \tag{7.1.20}$$

7.1.7. Let $x_1, \cdots, x_p$ be real scalar variables. Let

$$\begin{aligned} x_1 &= r \sin\theta_1 \\ x_j &= r\cos\theta_1 \cos\theta_2 \cdots \cos\theta_{j-1} \sin\theta_j, \;\; j = 2, 3, \cdots, p-1 \\ x_p &= r\cos\theta_1 \cos\theta_2 \cdots \cos\theta_{p-1} \end{aligned}$$

for $r > 0, -\frac{\pi}{2} < \theta_j \leq \frac{\pi}{2}, j = 1, \cdots, p-2, -\pi < \theta_{p-1} \leq \pi$. Then show that

$$\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p = r^{p-1} \left\{\prod_{j=1}^{p-1} |\cos\theta_j|^{p-j-1}\right\} \mathrm{d}r \wedge \mathrm{d}\theta_1 \wedge \cdots \wedge \mathrm{d}\theta_{p-1}. \tag{7.1.21}$$

7.1.8. Let $X = \frac{T}{|T|}$ where X and T are $p \times p$ lower triangular or upper triangular matrices of functionally independent real variables with positive diagonal elements. Then show that

$$\mathrm{d}X = (p-1)|T|^{-p(p+1)/2}\mathrm{d}T. \tag{7.1.22}$$

7.1.9. For real symmetric positive definite matrices X and Y show that

$$\lim_{t\to\infty} \left|I + \frac{XY}{t}\right|^{-t} = \mathrm{e}^{-\mathrm{tr}(XY)} = \lim_{t\to\infty} \left|I - \frac{XY}{t}\right|^{t}. \tag{7.1.23}$$

7.1.10. Let $X = (x_{ij}), W = (w_{ij})$ be lower triangular $p \times p$ matrices of distinct real variables with $x_{jj} > 0, w_{jj} > 0, j = 1, \cdots, p$, $\sum_{k=1}^{j} w_{jk}^2 = 1, j = 1, \cdots, p$. Let $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_p), \lambda_j > 0, j = 1, \cdots, p$, real and distinct where $\mathrm{diag}(\lambda_1, \cdots, \lambda_p)$ denotes a diagonal matrix with diagonal elements $\lambda_1, \cdots, \lambda_p$. Show that

$$X = DW \Rightarrow \mathrm{d}X = \left\{\prod_{j=1}^{p} \lambda_j^{j-1} w_{jj}^{-1}\right\} \mathrm{d}D \wedge \mathrm{d}W. \tag{7.1.24}$$

7.2. Real Matrix-variate Gamma and Related Functions

In the real scalar case the integral representation for a gamma function is the following:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x,\ \Re(\alpha) > 0. \tag{7.2.1}$$

Let X be a $p \times p$ real symmetric positive definite matrix and consider the integral

$$\Gamma_p(\alpha) = \int_{X=X'>0} |X|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(X)}\mathrm{d}X \tag{7.2.2}$$

where, when p=1 the equation (7.2.2) reduces to (7.2.1). We will evaluate (7.2.2) with the help of the Jacobian in (7.1.11). Let $T = (t_{ij})$ be a lower triangular matrix with positive diagonal elements. Put

$$X = TT' \Rightarrow \mathrm{d}X = 2^p\left\{\prod_{j=1}^{p} t_{jj}^{p+1-j}\right\}\mathrm{d}T,\ |TT'| = \prod_{j=1}^{p} t_{jj}^2,$$

$$\mathrm{tr}(X) = \mathrm{tr}(TT') = t_{11}^2 + (t_{21}^2 + t_{22}^2) + \cdots + (t_{p1}^2 + \cdots + t_{pp}^2).$$

Then,

$$\begin{aligned}\Gamma_p(\alpha) &= \int_T |TT'|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(TT')}\left\{2^p\prod_{j=1}^{p} t_{jj}^{p+1-j}\right\}\mathrm{d}T\\ &= \left\{\prod_{i>j}\int_{-\infty}^{\infty}\mathrm{e}^{-t_{ij}^2}\mathrm{d}t_{ij}\right\}\left\{\prod_{j=1}^{p}\int_0^\infty 2(t_{ij}^2)^{\alpha-\frac{j}{2}}\mathrm{e}^{-t_{jj}^2}\mathrm{d}t_{jj}\right\},\end{aligned}$$

since $t_{ij} > 0$ by definition and $t_{ij}, i \neq j$ could vary from $-\infty$ to ∞ subject to the condition that TT' is positive definite. But

$$\int_{-\infty}^{\infty}\mathrm{e}^{-t_{ij}^2}\mathrm{d}t_{ij} = \sqrt{\pi}\ \text{and}\ \int_0^\infty 2(t_{jj}^2)^{\alpha-\frac{j}{2}}\mathrm{e}^{-t_{jj}^2}\mathrm{d}t_{jj} = \Gamma\Big(\alpha - \frac{j-1}{2}\Big)$$

for $\Re(\alpha) > \frac{j-1}{2},\ j = 1, \cdots, p \Rightarrow \Re(\alpha) > \frac{p-1}{2}$. Note that in $\prod_{i>j}$ there are $1 + 2 + \cdots + p - 1 = p(p-1)/2$ factors and hence the final result is the following:

Definition 7.2.1. Real Matrix-variate gamma function.

$$\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{4}}\Gamma(\alpha)\Gamma\Big(\alpha - \frac{1}{2}\Big)\cdots\Gamma\Big(\alpha - \frac{p-1}{2}\Big),\ \Re(\alpha) > \frac{p-1}{2}. \tag{7.2.3}$$

We will call $\Gamma_p(\alpha)$ the real matrix-variate gamma. Observe that for $p = 1,\ \Gamma_p(\alpha)$ reduces to $\Gamma(\alpha)$.

7.2.1. Matrix-variate real gamma density

With the help of (7.2.2) we can create the real matrix-variate gamma density as follows, where X is a $p \times p$ real symmetric positive definite matrix:

$$f(X) = \begin{cases} \frac{1\times 1^{\alpha-\frac{p+1}{2}}}{\Gamma_p(\alpha)}e^{-\mathrm{tr}(X)}, X = X' > 0,\ \Re(\alpha) > \frac{p-1}{2} \\ 0,\ \text{elsewhere}\ . \end{cases} \tag{7.2.4}$$

If another parameter matrix is to be introduced then we obtain a gamma density with parameters $(\alpha, B), B = B' > 0$, as follows:

$$f_1(X) = \begin{cases} \frac{|B|^{\alpha}}{\Gamma_p(\alpha)}|X|^{\alpha-\frac{p+1}{2}}e^{-\mathrm{tr}(BX)}, X = X' > 0, B = B' > 0, \Re(\alpha) > \frac{p-1}{2} \\ 0, \text{elsewhere}. \end{cases} \tag{7.2.5}$$

As in the scalar case, two matrix random variables X and Y are said to be independently distributed if the joint density of X and Y is the product of their marginal densities. We will examine the densities of some functions of independently distributed matrix random variables. To this end we will introduce a few more functions.

Definition 7.2.2. A real matrix-variate beta function, denoted by $B_p(\alpha,\beta)$, is defined as

$$B_p(\alpha,\beta) = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)},\ \Re(\alpha) > \frac{p-1}{2}, \Re(\beta) > \frac{p-1}{2}. \tag{7.2.6}$$

The quantity in (7.2.6), analogous to the scalar case (p=1), is the real matrix-variate beta function. Let us try to obtain an integral representation for the real matrix-variate beta function of (7.2.6). Consider

$$\Gamma_p(\alpha)\Gamma_p(\beta) = \left[\int_{X=X'>0} |X|^{\alpha-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X)} \mathrm{d}X\right] \times \left[\int_{Y=Y'>0} |Y|^{\beta-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(Y)} \mathrm{d}Y\right]$$

where both X and Y are $p \times p$ matrices.

$$= \int \int |X|^{\alpha-\frac{p+1}{2}} |Y|^{\beta-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X+Y)} \mathrm{d}X \wedge \mathrm{d}Y.$$

Put $U = X + Y$ for a fixed X. Then

$$Y = U - X \Rightarrow |Y| = |U - X| = |U||I - U^{-\frac{1}{2}} X U^{-\frac{1}{2}}|$$

where, for convenience, $U^{\frac{1}{2}}$ is the symmetric positive definite square root of U. Observe that when two matrices A and B are nonsingular where AB and BA are defined, even if they do not commute,

$$|I - AB| = |I - BA|$$

and if $A = A' > 0$ and $B = B' > 0$ then

$$|I - AB| = |I - A^{\frac{1}{2}} B A^{\frac{1}{2}}| = |I - B^{\frac{1}{2}} A B^{\frac{1}{2}}|.$$

Now,

$$\Gamma_p(\alpha)\Gamma_p(\beta) = \int_U \int_X |U|^{\beta-\frac{p+1}{2}} |X|^{\alpha-\frac{p+1}{2}} |I - U^{-\frac{1}{2}} X U^{-\frac{1}{2}}|^{\beta-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(U)} \mathrm{d}U \wedge \mathrm{d}X.$$

Let $Z = U^{-\frac{1}{2}} X U^{-\frac{1}{2}}$ for fixed U. Then $\mathrm{d}X = |U|^{\frac{p+1}{2}} \mathrm{d}U$ by using (7.1.12). Now,

$$\Gamma_p(\alpha)\Gamma_p(\beta) = \int_Z |Z|^{\alpha-\frac{p+1}{2}} |I - Z|^{\beta-\frac{p+1}{2}} \mathrm{d}Z \int_{U=U'>0} |U|^{\alpha+\beta-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(U)} \mathrm{d}U.$$

Evaluation of the U-integral by using (7.2.2) yields $\Gamma_p(\alpha + \beta)$. Then we have

$$B_p(\alpha, \beta) = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha + \beta)} = \int_Z |Z|^{\alpha-\frac{p+1}{2}} |I - Z|^{\beta-\frac{p+1}{2}} \mathrm{d}Z.$$

Since the integral has to remain non-negative we have $Z = Z' > 0, I - Z > 0$. Therefore, one representation of a real matrix-variate beta function is the following, which is also called the type-1 beta integral.

$$B_p(\alpha,\beta) = \int_{0<Z=Z'<I} |Z|^{\alpha-\frac{p+1}{2}}|I-Z|^{\beta-\frac{p+1}{2}}\mathrm{d}Z, \Re(\alpha) > \frac{p-1}{2},\ \Re(\beta) > \frac{p-1}{2} \quad (7.2.7)$$

By making the transformation $V = I - Z$ note that α and β can be interchanged in the integral which also shows that $B_p(\alpha,\beta) = B_p(\beta,\alpha)$ in the integral representation also.

Let us make the following transformation in (7.2.7).

$$W = (I-Z)^{-\frac{1}{2}}Z(I-Z)^{-\frac{1}{2}} \Rightarrow W = (Z^{-1}-I)^{-1} \Rightarrow W^{-1} = Z^{-1} - I$$

$$\Rightarrow |W|^{-(p+1)}\mathrm{d}W = |Z|^{-(p+1)}\mathrm{d}Z \Rightarrow \mathrm{d}Z = |I+W|^{-(p+1)}\mathrm{d}W.$$

Under this transformation the integral in (7.2.7) becomes the following: Observe that

$$|Z| = |W||I+W|^{-1}, |I-Z| = |I+W|^{-1}.$$

$$B_p(\alpha,\beta) = \int_{W=W'>0} |W|^{\alpha-\frac{p+1}{2}}|I+W|^{-(\alpha+\beta)}\mathrm{d}W, \quad (7.2.8)$$

for $\Re(\alpha) > \frac{p-1}{2},\ \Re(\beta) > \frac{p-1}{2}$.

The representation in (7.2.8) is known as the *type-2 integral* for a real matrix-variate beta function. With the transformation $V = W^{-1}$ the parameters α and β in (7.2.8) will be interchanged. With the help of the type-1 and type-2 integral representations one can define the type-1 and type-2 beta densities in the real matrix-variate case.

Definition 7.2.3. Real matrix-variate type-1 beta density for the $p \times p$ real symmetric positive definite matrix X such that $X = X' > 0,\ I - X > 0$.

$$f_2(X) = \begin{cases} \frac{1}{B_p(\alpha,\beta)}|X|^{\alpha-\frac{p+1}{2}}|I-X|^{\beta-\frac{p+1}{2}} = 0 < X = X' < I,\ \Re(\alpha) > \frac{p-1}{2},\ \Re(\beta) > \frac{p-1}{2}, \\ 0,\ \text{elsewhere.} \end{cases} \quad (7.2.9)$$

Definition 7.2.4. Real matrix-variate type-2 beta density for the $p \times p$ real symmetric matrix X.

$$f_3(X) = \begin{cases} \frac{\Gamma_p(\alpha+\beta)}{\Gamma_p(\alpha)\Gamma_p(\beta)}|X|^{\alpha-\frac{p+1}{2}}|I+X|^{-(\alpha+\beta)}, X = X' > 0, \\ \qquad \Re(\alpha) > \frac{p-1}{2}, \;\; \Re(\beta) > \frac{p-1}{2} \\ 0, \text{ elsewhere.} \end{cases} \tag{7.2.10}$$

Example 7.2.1. Let X_1, X_2 be $p\times p$ matrix random variables, independently distributed as (7.2.4) with parameters α_1 and α_2 respectively. Let $U = X_1+X_2$, $V = (X_1+X_2)^{-\frac{1}{2}}X_1(X_1+X_2)^{-\frac{1}{2}}$, $W = X_2^{-\frac{1}{2}}X_1X_2^{-\frac{1}{2}}$. Evaluate the densities of U, V and W.

Solutions: The joint density of X_1 and X_2, denoted by $f(X_1, X_2)$, is available as the product of the marginal densities due to independence. That is,

$$f(X_1,X_2) = \frac{|X_1|^{\alpha_1-\frac{p+1}{2}}|X_2|^{\alpha_2-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(X_1+X_2)}}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_2)}, \; X_1 = X_1' > 0, \; X_2 = X_2' > 0,$$
$$\Re(\alpha_1) > \frac{p-1}{2}, \;\; \Re(\alpha_2) > \frac{p-1}{2}. \tag{7.2.11}$$

$$U = X_1 + X_2 \Rightarrow |X_2| = |U - X_1| = |U||I - U^{-\frac{1}{2}}X_1U^{-\frac{1}{2}}|.$$

Then the joint density of $(U, U_1) = (X_1 + X_2, X_1)$, the Jacobian is unity, is available as

$$f_1(U,U_1) = \frac{1}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_2)}|U_1|^{\alpha_1-\frac{p+1}{2}}|U|^{\alpha_2-\frac{p+1}{2}}|I - U^{-\frac{1}{2}}U_1U^{-\frac{1}{2}}|^{\alpha_2-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(U)}.$$

Put $U_2 = U^{-\frac{1}{2}}U_1U^{-\frac{1}{2}} \Rightarrow \mathrm{d}U_1 = |U|^{\frac{p+1}{2}}\mathrm{d}U_2$. Then the joint density of U and $U_2 = V$ is available as the following:

$$f_2(U,V) = \frac{1}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_2)}|U|^{\alpha_1+\alpha_2-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(U)}|V|^{\alpha_1-\frac{p+1}{2}}|I-V|^{\alpha_2-\frac{p+1}{2}}.$$

Since $f_2(U,V)$ is a product of two functions of U and V, $U = U' > 0, V = V' > 0$, $I - V > 0$ we see that U and V are independently distributed. The densities of U and V, denoted by $g_1(U)$, $g_2(V)$ are the following:

$$g_1(U) = c_1|U|^{\alpha_1+\alpha_2-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(U)}, U = U' > 0$$

and

$$g_2(V) = c_2|V|^{\alpha_1-\frac{p+1}{2}}|I-V|^{\alpha_2-\frac{p+1}{2}}, \; V = V' > 0, \; I - V > 0,$$

where c_1 and c_2 are the normalizing constants. But from the gamma density and type-1 beta density note that

$$c_1 = \frac{1}{\Gamma_p(\alpha_1+\alpha_2)},\ c_2 = \frac{\Gamma_p(\alpha_1+\alpha_2)}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_2)},\quad \Re(\alpha_1)>0,\quad \Re(\alpha_2)>0.$$

Hence U is gamma distributed with the parameter $(\alpha_1+\alpha_2)$ and V is type-1 beta distributed with the parameters α_1 and α_2 and further that U and V are independently distributed. For obtaining the density of $W = X_2^{-\frac{1}{2}} X_1 X_2^{-\frac{1}{2}}$ start with (7.2.11). Change (X_1, X_2) to (X_1, W) for fixed X_2. Then $\mathrm{d}X_1 = |X_2|^{\frac{p+1}{2}}\mathrm{d}W$. The joint density of X_2 and W, denoted by $f_{w,x_2}(W, X_2)$, is the following, observing that

$$\begin{aligned}\mathrm{tr}(X_1+X_2) &= \mathrm{tr}[X_2^{\frac{1}{2}}(I + X_2^{-\frac{1}{2}} X_1 X_2^{-\frac{1}{2}})X_2^{\frac{1}{2}}]\\ &= \mathrm{tr}[X_2^{\frac{1}{2}}(I+W)X_2^{\frac{1}{2}}] = \mathrm{tr}[(I+W)X_2]\\ &= \mathrm{tr}[(I+W)^{\frac{1}{2}}X_2(I+W)^{\frac{1}{2}}]\end{aligned}$$

by using the fact that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any two matrices where AB and BA are defined.

$$f_{w,x_2}(W, X_2) = \frac{1}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_2)}|W|^{\alpha_1-\frac{p+1}{2}}|X_2|^{\alpha_1+\alpha_2-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}[(I+W)^{\frac{1}{2}}X_2(I+W)^{\frac{1}{2}}]}.$$

Hence the marginal density of W, denoted by $g_w(W)$, is available by integrating out X_2 from $f_{w,x_2}(W, X_2)$. That is,

$$g_w(W) = \frac{1}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_2)}|W|^{\alpha_1-\frac{p+1}{2}}\int_{X_2=X_2'>0}|X_2|^{\alpha_1+\alpha_2-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}[(I+W)^{\frac{1}{2}}X_2(I+W)^{\frac{1}{2}}]}\mathrm{d}X_2.$$

Put $X_3 = (I+W)^{\frac{1}{2}}X_2(I+W)^{\frac{1}{2}}$ for fixed W, then $\mathrm{d}X_3 = |I+W|^{\frac{p+1}{2}}\mathrm{d}X_2$.
Then the integral becomes

$$\begin{aligned}&\int_{X_2=X_2'>0}|X_2|^{\alpha_1+\alpha_2-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}[(I+W)^{\frac{1}{2}}X_2(I+W)^{\frac{1}{2}}]}\mathrm{d}X_2\\ &= \Gamma_p(\alpha_1+\alpha_2)|I+W|^{-(\alpha_1+\alpha_2)}.\end{aligned}$$

Hence,

$$g_w(W) = \begin{cases} \frac{\Gamma_p(\alpha_1+\alpha_2)}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_2)}|W|^{\alpha_1-\frac{p+1}{2}}|I+W|^{-(\alpha_1+\alpha_2)}, W = W' > 0, \\ \qquad \Re(\alpha_1) > \frac{p-1}{2}, \quad \Re(\alpha_2) > \frac{p-1}{2} \\ 0, \text{elsewhere,} \end{cases}$$

which is a type-2 beta density with the parameters α_1 and α_2. Thus, W is real matrix-variate type-2 beta distributed.

7.3. The Laplace Transform in the Matrix Case

If $f(x_1, \cdots, x_k)$ is a scalar function of the real scalar variables $x_1, \cdots, x_k$ then the Laplace transform of f, with the parameters $t_1, \cdots, t_k$, is given by

$$L_f(t_1, \cdots, t_k) = \int_0^\infty \cdots \int_0^\infty \mathrm{e}^{-t_1x_1-\cdots-t_kx_k} f(x_1, \cdots, x_k)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k. \qquad (7.3.1)$$

If $f(X)$ is a real scalar function of the $p\times p$ real symmetric positive definite matrix X then the Laplace transform of $f(X)$ should be consistent with (7.3.1). When $X = X'$ there are only $p(p+1)/2$ distinct elements, either $x_{ij}'s, i \le j$ or $x_{ij}'s, i \ge j$. Hence what is needed is a linear function of all these variables. That is, in the exponent we should have the linear function $t_{11}x_{11} + (t_{21}x_{21} + t_{22}x_{22}) + \cdots + (t_{p1}x_{p1} + \cdots + t_{pp}x_{pp})$. Even if we take a symmetric matrix $T = (t_{ij}) = T'$ then the trace of TX,

$$\mathrm{tr}(TX) = t_{11}x_{11} + \cdots + t_{pp}x_{pp} + 2\sum_{i<j=1}^{p} t_{ij}x_{ij}.$$

Hence if we take a symmetric matrix of parameters t_{ij}'s such that

$$T^* = \begin{bmatrix} t_{11} & \frac{1}{2}t_{12} & \cdots & \frac{1}{2}t_{1p} \\ \frac{1}{2}t_{21} & t_{22} & \cdots & \frac{1}{2}t_{2p} \\ \vdots & & & \\ \frac{1}{2}t_{p1} & \frac{1}{2}t_{p2} & \cdots & t_{pp} \end{bmatrix}, \quad T^* = (t^*_{ij}) \Rightarrow t^*_{jj} = t_{jj}, t^*_{ij} = \frac{1}{2}t_{ij}, \quad i \neq j$$

then

$$\mathrm{tr}(T^*X) = t_{11}x_{11} + \cdots + t_{pp}x_{pp} + \sum_{i=1}^{p}\sum_{j=1,i>j}^{p} t_{ij}x_{ij}.$$

Hence the Laplace transform in the matrix case, for real symmetric positive definite matrix X, is defined with the parameter matrix T^*.

Definition 7.3.1. Laplace transform in the matrix case.

$$L_f(T^*) = \int_{X=X'>0} e^{-\mathrm{tr}(T^*X)} f(X)\mathrm{d}X, \tag{7.3.2}$$

whenever the integral is convergent.

Example 7.3.1. Evaluate the Laplace transform for the two-parameter gamma density in (7.2.5).

Solution: Here,

$$f(X) = \frac{|B|^\alpha}{\Gamma_p(\alpha)}|X|^{\alpha-\frac{p+1}{2}}e^{-\mathrm{tr}(BX)}, X = X' > 0, B = B' > 0, \quad \Re(\alpha) > \frac{p-1}{2}. \tag{7.3.3}$$

Hence the Laplace transform of f is the following:

$$L_f(T^*) = \frac{|B|^\alpha}{\Gamma_p(\alpha)} \int_{X=X'>0} |X|^{\alpha-\frac{p+1}{2}}e^{-\mathrm{tr}(T^*X)}e^{-\mathrm{tr}(BX)}\mathrm{d}X.$$

Note that since T^*, B and X are $p \times p$,

$$\mathrm{tr}(T^*X) + \mathrm{tr}(BX) = \mathrm{tr}[(B + T^*)X].$$

Thus for the integral to converge the exponent has to remain positive definite. Then the condition $B + T^* > 0$ is sufficient. Let $(B + T^*)^{\frac{1}{2}}$ be the symmetric positive definite square root of $B + T^*$. Then

$$\mathrm{tr}[(B + T^*)X] = \mathrm{tr}[(B + T^*)^{\frac{1}{2}}X(B + T^*)^{\frac{1}{2}}],$$

$$(B + T^*)^{\frac{1}{2}}X(B + T^*)^{\frac{1}{2}} = Y \Rightarrow \mathrm{d}X = |B + T^*|^{-\frac{p+1}{2}}\mathrm{d}Y$$

and

$$|X|^{\alpha-\frac{p+1}{2}}\mathrm{d}X = |B + T^*|^{-\alpha}|Y|^{\alpha-\frac{p+1}{2}}\mathrm{d}Y.$$

Hence,

$$\begin{aligned} L_f(T^*) &= \frac{|B|^\alpha}{\Gamma_p(\alpha)} \int_{Y=Y'>0} |B + T^*|^{-\alpha}|Y|^{\alpha-\frac{p+1}{2}}e^{-\mathrm{tr}(Y)}\mathrm{d}Y \\ &= |B|^\alpha|B + T^*|^{-\alpha} = |I + B^{-1}T^*|^{-\alpha}. \end{aligned} \tag{7.3.4}$$

Thus for known B and arbitrary T^*, (7.3.4) will uniquely determine (7.3.3) through the uniqueness of the inverse Laplace transform. The conditions for the uniqueness will not be discussed here. For some results in this direction see Mathai (1993, 1997) and the references therein.

7.3.1. A convolution property for Laplace transforms

Let $f_1(X)$ and $f_2(X)$ be two real scalar functions of the real symmetric positive definite matrix X and let $g_1(T^*)$ and $g_2(T^*)$ be their Laplace transforms. Let

$$f_3(X) = \int_{0<S=S'<X} f_1(X-S)f_2(S)\mathrm{d}S. \tag{7.3.5}$$

Then $g_1 g_2$ is the Laplace transform of $f_3(X)$.

This result can be established from the definition itself.

$$L_{f_3}(T^*) = \int_{X=X'>0} \mathrm{e}^{-\mathrm{tr}(T^*X)} f_3(X)\mathrm{d}X$$
$$= \int_{x>0}\int_{S<X} \mathrm{e}^{-\mathrm{tr}(T^*X)} f_1(X-S)f_2(S)\mathrm{d}S\,\mathrm{d}X.$$

Note that $\{S < X, X > 0\}$ is also equivalent to $\{X > S, S > 0\}$. Hence we may interchange the integrals. Then

$$L_{f_3}(T^*) = \int_{S>0} f_2(S)\left[\int_{X>S} \mathrm{e}^{-\mathrm{tr}(T^*X)} f_1(X-S)\mathrm{d}X\right]\mathrm{d}S.$$

Put $X - S = Y \Rightarrow X = Y + S$ and then

$$\begin{aligned} L_{f_3}(T^*) &= \int_{S>0} \mathrm{e}^{-\mathrm{tr}(T^*S)} f_2(S)\left[\int_{Y>0} \mathrm{e}^{-\mathrm{tr}(T^*Y)} f_1(Y)\mathrm{d}Y\right]\mathrm{d}S \\ &= g_2(T^*)g_1(T^*). \end{aligned}$$

Example 7.3.2. Using the convolution property for the Laplace transform and an integral representation for the real matrix-variate beta function show that

$$B_p(\alpha,\beta) = \Gamma_p(\alpha)\Gamma_p(\beta)/\Gamma_p(\alpha+\beta).$$

Solution: Let us start with the integral representation

$$\begin{aligned} B_p(\alpha,\beta) &= \int_{0<X<I} |X|^{\alpha-\frac{p+1}{2}}|I-X|^{\beta-\frac{p+1}{2}}\mathrm{d}X, \\ &\qquad \Re(\alpha) > \frac{p-1}{2}, \Re(\beta) > \frac{p-1}{2}. \end{aligned}$$

Consider the integral

$$\int_{0<U<X} |U|^{\alpha-\frac{p+1}{2}}|X-U|^{\beta-\frac{p+1}{2}}\mathrm{d}U = |X|^{\beta-\frac{p+1}{2}} \int_{0<U<X} |U|^{\alpha-\frac{p+1}{2}}$$
$$\times |I - X^{-\frac{1}{2}}UX^{-\frac{1}{2}}|^{\beta-\frac{p+1}{2}}\mathrm{d}U$$
$$= |X|^{\alpha+\beta-\frac{p+1}{2}} \int_{0<Y<I} |Y|^{\alpha-\frac{p+1}{2}}|I-Y|^{\beta-\frac{p+1}{2}}\mathrm{d}Y, Y = X^{-\frac{1}{2}}UX^{-\frac{1}{2}}.$$

Then

$$B_p(\alpha,\beta)|X|^{\alpha+\beta-\frac{p+1}{2}} = \int_{0<U<X} |U|^{\alpha-\frac{p+1}{2}}|X-U|^{\beta-\frac{p+1}{2}}\mathrm{d}U. \tag{7.3.6}$$

Take the Laplace transform on both sides to obtain the following:
On the left side,

$$B_p(\alpha,\beta)\int_{X>0} |X|^{\alpha+\beta-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(T^*X)}\mathrm{d}X = B_p(\alpha,\beta)|T^*|^{-(\alpha+\beta)}\Gamma_p(\alpha+\beta).$$

On the right side we get,

$$\int_{X>0} \mathrm{e}^{-\mathrm{tr}(T^*X)}\left[\int_{0<U<X} |U|^{\alpha-\frac{p+1}{2}}|X-U|^{\beta-\frac{p+1}{2}}\mathrm{d}U\right]\mathrm{d}X$$
$$= \Gamma_p(\alpha)\Gamma_p(\beta)|T^*|^{-(\alpha+\beta)} \text{ (by the convolution property in (7.3.5).)}$$

Hence

$$B_p(\alpha,\beta) = \Gamma_p(\alpha)\Gamma_p(\beta)/\Gamma_p(\alpha+\beta).$$

Example 7.3.3. Let $h(T^*)$ be the Laplace transform of $f(X)$, that is, $h(T^*) = L_f(T^*)$. Then show that the Laplace transform of $|X|^{-\frac{p+1}{2}}\Gamma_p(\frac{p+1}{2})f(X)$ is equivalent to $\int_{U>T^*} h(U)\mathrm{d}U$.

Solution: From (7.3.3) observe that for symmetric positive definite constant matrix B the following is an identity.

$$|B|^{-\alpha} = \frac{1}{\Gamma_p(\alpha)}\int_{X>0} |X|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(BX)}\mathrm{d}X, \Re(\alpha) > \frac{p-1}{2}. \tag{7.3.7}$$

Then we can replace $|X|^{-\frac{p+1}{2}}\Gamma_p(\frac{p+1}{2})$ by an equivalent integral.

$$|X|^{-\frac{p+1}{2}}\Gamma_p\left(\frac{p+1}{2}\right) \equiv \int_{Y>0} |Y|^{\frac{p+1}{2}-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(XY)}\mathrm{d}Y = \int_{Y>0} \mathrm{e}^{-\mathrm{tr}(XY)}\mathrm{d}Y.$$

Then the Laplace transform of $|X|^{-\frac{p+1}{2}}\Gamma_p\left(\frac{p+1}{2}\right)f(X)$ is given by,

$$\begin{aligned}&\int_{X>0} e^{-\mathrm{tr}(T^*X)}f(X)[\int_{Y>0} e^{-\mathrm{tr}(YX)}\mathrm{d}Y]\mathrm{d}X\\ &\quad=\int_{X>0}\int_{Y>0} e^{-\mathrm{tr}[(T^*+Y)X]}f(X)\mathrm{d}Y\mathrm{d}X.\ \ (\text{Put}\ \ T^*+Y=U\Rightarrow U>T^*)\\ &\quad=\int_{Y>0} h(T^*+Y)\mathrm{d}Y=\int_{U>T^*} h(U)\mathrm{d}U.\end{aligned}$$

Example 7.3.4. For $X>B, B=B'>0$ and $\nu>-1$ show that the Laplace transform of $|X-B|^{\nu}$ is $|T|^{-(\nu+\frac{p+1}{2})}e^{-\mathrm{tr}(T^*B)}\Gamma_p(\nu+\frac{p+1}{2})$.

Solution: Laplace transform of $|X-B|^{\nu}$ with parameter matrix T^* is given by,

$$\begin{aligned}\int_{X>B}|X-B|^{\nu}e^{-\mathrm{tr}(T^*X)}\mathrm{d}X&=e^{-\mathrm{tr}(BT^*)}\int_{Y>0}|Y|^{\nu}e^{-\mathrm{tr}(T^*Y)}\mathrm{d}Y, Y=X-B\\ &=e^{-\mathrm{tr}(BT^*)}\ \Gamma_p\left(\nu+\frac{p+1}{2}\right)|T^*|^{-(\nu+\frac{p+1}{2})}\end{aligned}$$

(by writing $\nu=\nu+\frac{p+1}{2}-\frac{p+1}{2}$) for $\nu+\frac{p+1}{2}>\frac{p-1}{2}\Rightarrow\nu>-1$.

Exercises 7.3.

7.3.1. By using the process in Example 7.3.3, or otherwise, show that the Laplace transform of $[\Gamma_p(\frac{p+1}{2})|X|^{-\frac{p+1}{2}}]^n f(X)$ can be written as

$$\int_{W_1>T^*}\int_{W_2>W_1}\cdots\int_{W_n>W_{n-1}} h(W_n)\mathrm{d}W_1\wedge\cdots\wedge\mathrm{d}W_n$$

where $h(T^*)$ is the Laplace transform of f(X).

7.3.2. Show that the Laplace transform of $|X|^n$ is $|T^*|^{-n-\frac{p+1}{2}}\Gamma_p(n+\frac{p+1}{2})$ for $n>-1$.

7.3.3. If the $p\times p$ real matrix random variable X has a type-1 beta density with parameters (α_1,α_2) then show that

(i) U $=(I-X)^{-\frac{1}{2}}X(I-X)^{-\frac{1}{2}}\sim$ type-2 beta (α_1,α_2)

(ii) V $=X^{-1}-I\sim$ type-2 beta (α_2,α_1)

where " $\sim$ " indicates "distributed as", and the parameters are given in the brackets.

7.3.4. If the $p \times p$ real symmetric positive definite matrix random variable X has a type-2 beta density with parameters α_1 and α_2 then show that

$$\text{(i) } U = X^{-1} \sim \text{ type-2 beta } (\alpha_2, \alpha_1)$$

$$\text{(ii) } V = (I + X)^{-1} \sim \text{ type-1 beta } (\alpha_2, \alpha_1)$$

$$\text{(iii) } W = (I + X)^{-\frac{1}{2}} X (I + X)^{-\frac{1}{2}} \sim \text{ type-1 beta } (\alpha_1, \alpha_2).$$

7.3.5. If the Laplace transform of $f(X)$ is $g(T^*) = L_{T^*}(f(X))$, where X is real symmetric positive definite and $p \times p$ then show that

$$\Delta^n g(T^*) = L_{T^*}(|X|^n f(X)), \quad \Delta = (-1)^p \left| \frac{\partial}{\partial T^*} \right|$$

where $|\frac{\partial}{\partial T^*}|$ means that first the partial derivatives with respect to t_{ij}'s for all i and j are taken, then written in the matrix form and then the determinant is taken, where $T^* = (t^*_{ij})$.

7.4. Hypergeometric Functions with Matrix Argument

There are essentially three approaches available in the literature for defining a hypergeometric function of matrix argument. One approach due to Bochner (1951) and Herz (1955) is through Laplace and inverse Laplace transforms. Under this approach, a hypergeometric function is defined as the function satisfying a pair of integral equations, and explicit forms are available for ${}_0F_0$ and ${}_1F_0$. Another approach is available from James (1961) and Constantine (1963) through a series form involving zonal polynomials. Theoretically, explicit forms are available for general parameters or for a general ${}_pF_q$ but due to the difficulty in computing higher order zonal polynomials, computations are feasible only for small values of p and q. For a detailed discussion of zonal polynomials see Mathai, Provost and Hayakawa (1995). The third approach is due to Mathai (1978, 1993) with the help of a generalized matrix transform or M-transform. Through this definition a hypergeometric function is defined as a class of functions satisfying a certain integral equation. This definition is the one most suited for studying various properties of hypergeometric functions. The series form is least suited for this purpose. All these definitions are introduced for symmetric functions in the sense that

$$f(X) = f(X^{'}) = f(QQ^{'}X) = f(Q^{'}XQ) = f(D),\ D = \mathrm{diag}(\lambda_1, ..., \lambda_p).$$

If X is $p \times p$ and symmetric then there exists an orthonormal matrix Q, that is, $QQ^{'} = I, Q^{'}Q = I$ such that $Q^{'}XQ = \mathrm{diag}(\lambda_1, ..., \lambda_p)$ where $\lambda_1, ..., \lambda_p$ are the eigenvalues of X. Thus, f(X), a scalar function of the $p(p+1)/2$ functionally independent elements in X, is essentially a function of the p variables $\lambda_1, ..., \lambda_p$.

7.4.1. Hypergeometric function through Laplace transform

Let ${}_rF_s(a_1, ..., a_r; b_1, ..., b_s; Z)$ be the hypergeometric function of the matrix argument Z to be defined, $Z = Z^{'}$. Consider the following pair of Laplace and inverse Laplace transforms.

$$\begin{aligned}&{}_{r+1}F_s(a_1, ..., a_r, c; b_1, ..., b_s; -\wedge^{-1})|\wedge|^{-c}\\&= \frac{1}{\Gamma_p(c)} \int_{U=U'>0} \mathrm{e}^{-\mathrm{tr}(\wedge U)} {}_rF_s(a_1, ..., a_r; b_1, ..., b_s; -U)|U|^{c-\frac{p+1}{2}} \mathrm{d}U \qquad (7.4.1)\end{aligned}$$

and

$$\begin{aligned}&{}_rF_{s+1}(a_1, ..., a_r; b_1, ..., b_r, c; -\wedge)|\wedge|^{c-\frac{p+1}{2}}\\&= \frac{\Gamma_p(c)}{(2\pi i)^{p(p+1)/2}} \int_{\Re(Z)=X>X_0} \mathrm{e}^{\mathrm{tr}(\wedge Z)} {}_rF_s(a_1, ..., a_r; b_1, ..., b_s; -Z^{-1})|Z|^{-c}\mathrm{d}Z \quad (7.4.2)\end{aligned}$$

where $Z = X + iY,\ i = \sqrt{-1},\ X = X^{'} > 0$, and X and Y belong to the class of symmetric matrices with the non-diagonal elements weighted by $\frac{1}{2}$. The function ${}_rF_s$ satisfying (7.4.1) and (7.4.2) can be shown to be unique under certain conditions and that function is defined as the hypergeometric function of matrix argument $\wedge$, according to this definition.

Then by taking ${}_0F_0(;;-\wedge) = \mathrm{e}^{-\mathrm{tr}(\wedge)}$ and by using the convolution property of the Laplace transform and equations (7.4.1) and (7.4.2) one can systematically build up. The Bessel function ${}_0F_1$ for matrix argument is defined by Herz (1955). Thus we can go from ${}_0F_0$ to ${}_1F_0$ to ${}_0F_1$ to ${}_1F_1$ to ${}_2F_1$ and so on to a general ${}_pF_q$.

Example 7.4.1. Obtain an explicit form for ${}_1F_0$ from the above definition by using ${}_0F_0$ $(;;-U) = \mathrm{e}^{-\mathrm{tr}(U)}$.

Solution: From (7.4.1)

$$\frac{1}{\Gamma_p(c)}\int_{U=U'>0}|U|^{c-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(\wedge U)}{}_0F_0(;;-U)\mathrm{d}U$$
$$=\frac{1}{\Gamma_p(c)}\int_{U>0}|U|^{c-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}[(I+\wedge)U]}\mathrm{d}U=|I+\wedge|^{-c},$$

since

$${}_0F_0(;;-U)=\mathrm{e}^{-\mathrm{tr}(U)}.$$

But

$$|I+\wedge|^{-c}=|\wedge|^{-c}|I+\wedge^{-1}|^{-c}.$$

Then from (7.4.1)

$${}_1F_0(c;;-\wedge^{-1})=|I+\wedge^{-1}|^{-c}$$

which is an explicit representation.

7.4.2. Hypergeometric function through zonal polynomials

Zonal polynomials are certain symmetric functions in the eigenvalues of the $p\times p$ matrix Z. They are denoted by $C_K(Z)$ where K represents the partition of the positive integer k, $K=(k_1,...,k_p)$ with $k_1+\cdots+k_p=k$. When Z is 1×1 then $C_K(z)=z^k$. Thus, $C_K(Z)$ can be looked upon as a generalization of z^k in the scalar case. For details see Mathai, Provost and Hayakawa (1995). In terms of $C_K(Z)$ we have the representation for a

$${}_0F_0(;;Z)=\mathrm{e}^{\mathrm{tr}(Z)}=\sum_{k=0}^{\infty}\frac{(\mathrm{tr}(Z))^k}{k!}=\sum_{k=0}^{\infty}\sum_{K}\frac{C_K(Z)}{k!}. \tag{7.4.3}$$

The binomial expansion will be the following:

$${}_1F_0(\alpha;;Z)=\sum_{k=0}^{\infty}\sum_{K}\frac{(\alpha)_K C_K(Z)}{k!}=|I-Z|^{-\alpha}, \tag{7.4.4}$$

for $0<Z<I$, where,

$$(\alpha)_K=\prod_{j=1}^{p}\left(\alpha-\frac{j-1}{2}\right)_{k_j}, K=(k_1,...,k_p), k_1+\cdots+k_p=k. \tag{7.4.5}$$

In terms of zonal polynomials a hypergeometric series is defined as follows:

$$ {}_pF_q(a_1,...,a_p;b_1,...,b_q;Z) = \sum_{k=0}^{\infty}\sum_K \frac{(a_1)_K\cdots(a_p)_K}{(b_1)_K\cdots(b_q)_K}\frac{C_K(Z)}{k!}. \tag{7.4.6}$$

For (7.4.6) to be defined, none of the denominator factors is equal to zero, $q \geq p$, or $q = p+1$ and $0 < Z < I$. For other details see Constantine (1963). In order to study properties of a hypergeometric function with the help of (7.4.6) one needs the Laplace and inverse Laplace transforms of zonal polynomials. These are the following:

$$\int_{X=X'>0}|X|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(XZ)}C_K(XT)\mathrm{d}X = |Z|^{-\alpha}C_K(TZ^{-1})\Gamma_p(\alpha,K) \tag{7.4.7}$$

where

$$\Gamma_p(\alpha,K) = \pi^{p(p-1)/4}\prod_{j=1}^{p}\Gamma\left[\alpha+k_j-\frac{j-1}{2}\right] = \Gamma_p(\alpha)(\alpha)_K. \tag{7.4.8}$$

$$= \frac{1}{(2\pi i)^{p(p+1)/2}}\int_{\Re(Z)=X>X_0}\mathrm{e}^{\mathrm{tr}(SZ)}|Z|^{-\alpha}C_K(Z)\mathrm{d}Z$$

$$= \frac{1}{\Gamma_p(\alpha,K)}|S|^{\alpha-\frac{p+1}{2}}C_K(S), i = \sqrt{-1} \tag{7.4.9}$$

for $Z = X+iY$, $X = X' > 0$, X and Y are symmetric and the nondiagonal elements are weighted by $\frac{1}{2}$. If the non-diagonal elements are not weighted then the left side in (7.4.9) is to be multiplied by $2^{p(p-1)/2}$. Further,

$$\int_{0<X<I}|X|^{\alpha-\frac{p+1}{2}}|I-X|^{\beta-\frac{p+1}{2}}C_K(TX)\mathrm{d}X = \frac{\Gamma_p(\alpha,K)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta,K)}C_K(T)$$

$$\Re(\alpha) > \frac{p-1}{2},\ \Re(\beta) > \frac{p-1}{2}. \tag{7.4.10}$$

Example 7.4.2. By using zonal polynomials establish the following results:

$$\begin{aligned} {}_2F_1(a,b;c;X) &= \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)} \\ &\times \int_{0<\wedge<I}|\wedge|^{a-\frac{p+1}{2}}|I-\wedge|^{c-a-\frac{p+1}{2}}|I-\wedge X|^{-b}\mathrm{d}\wedge \end{aligned} \tag{7.4.11}$$

for $\Re(a) > \frac{p-1}{2}$, $\Re(c-a) > \frac{p-1}{2}$.

Solution: Expanding $|I - \wedge X|^{-b}$ in terms of zonal polynomials and then integrating term by term the right side reduces to the following:

$$|I - \wedge X|^{-b} = \sum_{k=0}^{\infty}\sum_{K}(b)_K \frac{C_K(\wedge X)}{k!} \quad \text{for } 0 < \wedge X < I$$

and

$$\int_{0<\wedge<I} |\wedge|^{a-\frac{p+1}{2}} |I - \wedge|^{c-a-\frac{p+1}{2}} C_K(\wedge X)\mathrm{d}\wedge = \frac{\Gamma_p(a,K)\Gamma_p(c-a)}{\Gamma_p(c,K)} C_K(X)$$

by using (7.4.10). But

$$\frac{\Gamma_p(a,K)\Gamma_p(c-a)}{\Gamma_p(c,K)} = \frac{\Gamma_p(a)\Gamma_p(c-a)}{\Gamma_p(c)} \frac{(a)_K}{(c)_K}.$$

Substituting these back, the right side becomes

$$\sum_{k=0}^{\infty}\sum_{K} \frac{(a)_K(b)_K}{(c)_K} \frac{C_K(X)}{k!} = {}_2F_1(a,b;c;X).$$

This establishes the result.

Example 7.4.3. Establish the result

$${}_2F_1(a,b;c;I) = \frac{\Gamma_p(c)\Gamma_p(c-a-b)}{\Gamma_p(c-a)\Gamma_p(c-b)} \tag{7.4.12}$$

for $\Re(c-a-b) > \frac{p-1}{2}$, $\Re(c-a) > \frac{p-1}{2}$, $\Re(c-b) > \frac{p-1}{2}$.

Solution: In (7.4.11) put X=I, combine the last factor on the right with the previous factor and integrate out with the help of a matrix-variate type-1 beta integral.

Uniqueness of the ${}_pF_q$ through zonal polynomials, as given in (7.4.6), is established by appealing to the uniqueness of the function defined through the Laplace and inverse Laplace transform pair in (7.4.1) and (7.4.2), and by showing that (7.4.6) satisfies (7.4.1) and (7.4.2).

The next definition, introduced by Mathai in a series of papers is through a special case of Weyl's fractional integral.

7.4.3. Hypergeometric functions through M-transforms

Consider the class of $p \times p$ real symmetric definite matrices and the null matrix O. Any member of this class will be either positive definite or negative definite or null. Let α be a complex parameter such that $\Re(\alpha) > \frac{p-1}{2}$. Let $f(S)$ be a scalar symmetric function in the sense $f(AB) = f(BA)$ for all A and B when AB and BA are defined. Then the M-transform of $f(S)$, denoted by $M_\alpha(f)$, is defined as

$$M_\alpha(f) = \int_{U=U'>0} |U|^{\alpha-\frac{p+1}{2}} f(U)\mathrm{d}U \tag{7.4.13}$$

Some examples of symmetric functions are $\mathrm{e}^{\pm\mathrm{tr}(S)}$, $|I \pm S|^\beta$ for nonsingular $p \times p$ matrices A and B such that,

$$\mathrm{e}^{\pm\mathrm{tr}(AB)} = \mathrm{e}^{\pm\mathrm{tr}(BA)};\ |I \pm AB|^\beta = |I \pm BA|^\beta.$$

Is it possible to recover $f(U)$, a function of $p(p+1)/2$ elements in $U = (u_{ij})$ or a function of p eigenvalues of U, that is a function of p variables, from $M_\alpha(t)$ which is a function of one parameter α? In a normal course the answer is in the negative. But due to the properties that are seen, it is clear that there exists a set of sufficient conditions by which $M_\alpha(f)$ will uniquely determine $f(U)$. It is easy to note that the class of functions defined through (7.4.13) satisfy the pair of integral equations (7.4.1) and (7.4.2) defining the unique hypergeometric function.

A hypergeometric function through M-transform is defined as a class of functions ${}_rF_s^*$ satisfying the following equation:

$$\int_{X=X'>0} |X|^{\alpha-\frac{p+1}{2}} {}_rF_s{}^*(a_1, ..., a_p; b_1, ..., b_q; -X)\mathrm{d}X$$
$$= \frac{\{\prod_{j=1}^s \Gamma_p(b_j)\}}{\{\prod_{j=1}^r \Gamma_p(a_j)\}} \frac{\{\prod_{j=1}^r \Gamma_p(a_j-\rho)\}}{\{\prod_{j=1}^s \Gamma_p(b_j-\rho)\}} \Gamma_p(\rho) \tag{7.4.14}$$

where ρ is an arbitrary parameter such that the gammas exist.

Example 7.4.4. Re-establish the result

$$L_T(|X-B|^\nu) = \Gamma_p\left(\nu + \frac{p+1}{2}\right)|T|^{-(\nu+\frac{p+1}{2})}\mathrm{e}^{-\mathrm{tr}(TB)} \tag{7.4.15}$$

by using M-transforms.

Solution: We will show that the M-transforms on both sides of (7.4.15) are one and the same. Taking the M-transform of the left-side, with respect to the parameter ρ, we have,

$$\begin{aligned}\int_{T>0}|T|^{\rho-\frac{p+1}{2}}\{L_T(|X-B|^{\nu}\}\mathrm{d}T &= \int_{T>0}|T|^{\rho-\frac{p+1}{2}}\left[\int_{X>B}|X-B|^{\nu}\mathrm{e}^{-\mathrm{tr}(TX)}\mathrm{d}X\right]\mathrm{d}T\\ &= \int_{T>0}|T|^{\rho-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(TB)}\left[\int_{Y>0}|Y|^{\nu}\mathrm{e}^{-\mathrm{tr}(TY)}\mathrm{d}Y\right]\mathrm{d}T.\end{aligned}$$

Noting that $\nu = \nu + \frac{p+1}{2} - \frac{p+1}{2}$ the Y-integral gives $|T|^{-\nu-\frac{p+1}{2}}\Gamma_p(\nu+\frac{p+1}{2})$. Then the T-integral gives

$$M_{\rho}(\text{left-side}) = \Gamma_p\left(\nu+\frac{p+1}{2}\right)\Gamma_p\left(\rho-\nu-\frac{p+1}{2}\right)|B|^{-\rho+\nu+\frac{p+1}{2}}.$$

M-transform of the right side gives,

$$\begin{aligned}M_{\rho}(\text{right-side}) &= \int_{T>0}|T|^{\rho-\frac{p+1}{2}}\left\{\Gamma_p\left(\nu+\frac{p+1}{2}\right)|T|^{-(\nu+\frac{p+1}{2})}\mathrm{e}^{-\mathrm{tr}(TB)}\right\}\mathrm{d}T\\ &= \Gamma_p\left(\nu+\frac{p+1}{2}\right)\Gamma_p\left(\rho-\nu-\frac{p+1}{2}\right)|B|^{-\rho+\nu+\frac{p+1}{2}}\end{aligned}$$

The two sides have the same M-transform.

Starting with ${}_0F_0(;;X) = \mathrm{e}^{\mathrm{tr}(X)}$, we can build up a general ${}_pF_q$ by using the M-transform and the convolution form for M-transforms, which will be stated next.

7.4.4. A convolution theorem for M-transforms

Let $f_1(U)$ and $f_2(U)$ be two symmetric scalar functions of the $p\times p$ real symmetric positive definite matrix U, with M-transforms $M_{\rho}(f_1) = g_1(\rho)$ and $M_{\rho}(f_2) = g_2(\rho)$ respectively. Let

$$f_3(S) = \int_{U>0}|U|^{B}f_1(U^{\frac{1}{2}}SU^{\frac{1}{2}})f_2(U)\mathrm{d}U \tag{7.4.16}$$

then the M-transform of f_3 is given by,

$$M_\rho(f_3) = g_1(\rho)g_2\left(\beta - \rho + \frac{p+1}{2}\right). \tag{7.4.17}$$

The result can be easily established from the definition itself by interchanging the integrals.

Example 7.4.5. Show that

$${}_1F_1(a;c;-\wedge) = \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)}\int_{0<U<I}|U|^{a-\frac{p+1}{2}}|I-U|^{c-a-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(\wedge U)}\mathrm{d}U. \tag{7.4.18}$$

Solution: We will establish this by showing that both sides have the same M-transforms. From the definition in (7.4.14) the M-transform of the left side with respect to the parameter ρ is given by the following:

$$\begin{aligned}
M_\rho(\text{left-side}) &= \int_{\wedge=\wedge'>0} |\wedge|^{\rho-\frac{p+1}{2}} \; {}_1F_1(a;c;-\wedge)\mathrm{d}\wedge \\
&= \left[\frac{\Gamma_p(a-\rho)}{\Gamma_p(c-\rho)}\Gamma_p(\rho)\right]\frac{\Gamma_p(c)}{\Gamma_p(a)}. \\
M_\rho(\text{right-side}) &= \int_{\wedge>0} |\wedge|^{\rho-\frac{p+1}{2}}\left\{\frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)}\right. \\
&\quad \times \left.\int_{0<U<I}|U|^{a-\frac{p+1}{2}}|I-U|^{c-a-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(\wedge U)}\mathrm{d}U\right\}\mathrm{d}\wedge .
\end{aligned}$$

Take,

$$f_1(U) = \mathrm{e}^{-\mathrm{tr}(U)} \text{ and } f_2(U) = |U|^{a-\frac{p+1}{2}}|I-U|^{c-a-\frac{p+1}{2}}.$$

Then

$$M_\rho(f_1) = g_1(\rho) = \int_{U>0} |U|^{\rho-\frac{p+1}{2}} e^{-\mathrm{tr}(U)} \mathrm{d}U = \Gamma_p(\rho), \Re(\rho) > \frac{p+1}{2}.$$

$$M_\rho(f_2) = g_2(\rho) = \int_{U>0} |U|^{\rho-\frac{p+1}{2}} |U^{a-\frac{p+1}{2}}|I-U|^{c-a-\frac{p+1}{2}} \mathrm{d}U$$

$$= \frac{\Gamma_p(a+\rho-\frac{p+1}{2})\Gamma_p(c-a)}{\Gamma_p(c+s-\frac{p+1}{2})}, \Re(c-a) > \frac{p-1}{2},$$

$$\Re(a+\rho) > p, \Re(c+\rho) > p.$$

Taking f_3 in (7.4.16) as the second integral on the right above we have

$$M_\rho(\text{right-side}) = \left\{\frac{\Gamma_p(c)}{\Gamma_p(a)}\right\} \Gamma_p(\rho) \frac{\Gamma_p(a-\rho)}{\Gamma_p(c-\rho)} = M_\rho(\text{left-side}).$$

Hence the result.

Almost all properties, analogous to the ones in the scalar case for hypergeometric functions, can be established by using the M-transform technique very easily. These can then be shown to be unique , if necessary, through the uniqueness of Laplace and inverse Laplace transform pair. Theories for functions of several matrix arguments, Dirichlet integrals, Dirichlet densities, their extensions, Appell's functions, Lauricella functions, and the like, are available. Then all these real cases are also extended to complex cases as well. For details see Mathai (1997). Problems involving scalar functions of matrix argument, real and complex cases, are still being worked out and applied in many areas such as statistical distribution theory, econometrics, quantum mechanics and engineering areas. Since the aim in this brief note is only to introduce the subject matter, more details will not be given here.

Exercises 7.4.

7.4.1. Show that for $\wedge = \wedge' > 0$ and $p \times p$,

$${}_1F_1(a; c; -\wedge) = e^{-\mathrm{tr}(\wedge)} {}_1F_1(c-a; c; \wedge).$$

7.4.2. For $p \times p$ real symmetric positive definite matrices $\wedge$ and $\vee$ show that

$$
\begin{aligned}
{}_1F_1(a;c;-\wedge) &= \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)}|\wedge|^{-(c-\frac{p+1}{2})}\int_{0<\vee<\wedge} e^{-\mathrm{tr}(\vee)} \\
&\quad \times|\vee|^{a-\frac{p+1}{2}}|\wedge-\vee|^{c-a-\frac{p+1}{2}}\mathrm{d}V.
\end{aligned}
$$

7.4.3. Show that for ϵ a scalar and A a $p\times p$ matrix with p finite

$$
\lim_{\epsilon\to 0}|I+\epsilon A|^{-\frac{1}{\epsilon}} = \lim_{\epsilon\to\infty}|I+\frac{A}{\epsilon}|^{-\epsilon} = e^{-\mathrm{tr}(A)}.
$$

7.4.4. Show that

$$
\begin{aligned}
\lim_{a\to\infty} {}_1F_1(a;c;-\frac{Z}{a}) &= \lim_{\epsilon\to 0} {}_1F_1\left(\frac{1}{\epsilon};c;-\epsilon Z\right) \\
&= {}_0F_1(;c;-Z).
\end{aligned}
$$

7.4.5. Show that

$$
{}_1F_1(a;c;-\wedge) = \frac{\Gamma_p(c)}{(2\pi i)^{\frac{p(p+1)}{2}}}\int_{\Re(Z)=X>X_0} e^{\mathrm{tr}(Z)}|Z|^{-c}|I+\wedge Z^{-1}|^{-a}\mathrm{d}Z.
$$

7.4.6. Show that

$$
{}_2F_1(a,b;c;X) = |I-X|^{-\beta}{}_2F_1(c-a,b;c;-X(I-X)^{-1}).
$$

7.4.7. For $\Re(s)>\frac{p-1}{2}$, $\Re(b-s)>\frac{p-1}{2}$, $\Re(c-a-s)>\frac{p-1}{2}$, show that

$$
\begin{aligned}
&\int_{0<X<I}|X|^{s-\frac{p+1}{2}}|I-X|^{b-s-\frac{p+1}{2}}{}_2F_1(a,b;c;X)\mathrm{d}X \\
&= \frac{\Gamma_p(c)\Gamma_p(s)\Gamma_p(b-s)\Gamma_p(c-a-s)}{\Gamma_p(b)\Gamma_p(c-a)\Gamma_p(c-s)}.
\end{aligned}
$$

7.4.8. Defining the Bessel function $A_r(S)$ with $p\times p$ real symmetric positive definite matrix argument S, as

$$
A_r(S) = \frac{1}{\Gamma_p(r+\frac{p+1}{2})}{}_0F_1(;r+\frac{p+1}{2};-S), \qquad (7.4.19)
$$

show that

$$\int_{S>0} |S|^{\delta-\frac{p+1}{2}} A_r(S) e^{-\mathrm{tr}(\wedge S)} \mathrm{d}S = \frac{\Gamma_p(\delta)}{\Gamma_p(r+\frac{p+1}{2})} |\wedge|^{-\delta} {}_1F_1\left(\delta; r+\frac{p+1}{2}; -\wedge^{-1}\right).$$

7.4.9. If

$$M(\alpha,\beta;A) = \int_{X=X'>0} |X|^{\alpha-\frac{p+1}{2}} |I+X|^{\beta-\frac{p+1}{2}} e^{-\mathrm{tr}(AX)} \mathrm{d}X,$$
$$\Re(\alpha) > \frac{p-1}{2}, A = A' > 0$$

then show that

$$\int_{X>0} |X+A|^{\nu} e^{-\mathrm{tr}(TX)} \mathrm{d}X = |A|^{\nu+\frac{p+1}{2}} M\left(\frac{p+1}{2}, \nu+\frac{p+1}{2}; A^{\frac{1}{2}} T A^{\frac{1}{2}}\right).$$

7.4.10. If Whittaker function W is defined as

$$\int_{Z>0} |Z|^{\mu-\frac{p+1}{2}} |I+Z|^{\nu-\frac{p+1}{2}} e^{-\mathrm{tr}(AZ)} \mathrm{d}Z$$
$$= |A|^{-\frac{\mu+\nu}{2}} \Gamma_p(\mu) e^{\frac{1}{2}\mathrm{tr}(A)} W_{\frac{1}{2}(\nu-\mu),\frac{1}{2}(\nu+\mu-\frac{(p+1)}{2})}(A)$$

then show that

$$\int_{X>U} |X+B|^{2\alpha-\frac{p+1}{2}} |X-U|^{2q-\frac{p+1}{2}} e^{-\mathrm{tr}(MX)} \mathrm{d}X$$
$$= |U+B|^{\alpha+q-\frac{p+1}{2}} |M|^{-(\alpha+q)} e^{\frac{1}{2}\mathrm{tr}[(B-U)M]} \Gamma_p(2q)$$
$$\times W_{(\alpha-q),(\alpha+q-\frac{(p+1)}{4})}(S), S = (U+B)^{\frac{1}{2}} M (U+B)^{\frac{1}{2}}.$$

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