LECTURE NOTES

SPECIAL FUNCTIONS AND FUNCTIONS OF MATRIX ARGUMENT: RECENT DEVELOPMENTS AND RECENT APPLICATIONS IN STATISTICS AND ASTROPHYSICS

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SPECIAL FUNCTIONS AND FUNCTIONS OF MATRIX ARGUMENT: RECENT DEVELOPMENTS AND RECENT APPLICATIONS IN STATISTICS AND ASTROPHYSICS

LECTURE NOTES

of the
THIRD S.E.R.C. SCHOOL ON SPECIAL FUNCTIONS AND
FUNCTIONS OF MATRIX ARGUMENT:
RECENT DEVELOPMENTS AND RECENT APPLICATIONS
IN STATISTICS AND ASTROPHYSICS
14th March 2005 to 15th April 2005

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Notes compiled by
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This is publication No.32 of the Publications Series of the Centre. This is compiled from the lectures to be given at the third S.E.R.C. School on Special Functions and Functions of Matrix Argument: Recent Developments and Recent Applications in Statistics and Astrophysics, sponsored by the Department of Science and Technology, New Delhi, India, and conducted by this Centre during the period 14th March 2005 to 15th April 2005 with Dr. A.M. Mathai as the Director and Dr. K.K. Jose as the Co-Director of the School.

The Centre would like to thank the Department of Science and Technology, New Delhi, India for financial assistance which made the School as well as this publication possible and Dr. B.D. Acharya, Advisor to Government of India and the Mathematical Sciences Office of DST, Delhi for the constant encouragement, and whose enthusiasm and initiative are mainly responsible for conducting this School and for bringing out this publication.

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PREFACE

The first S.E.R.C. School on Special Functions was sponsored by DST, Delhi and conducted by CMS for six weeks in 1995. The second S.E.R.C. School on Special Functions and Functions of Matrix Argument was sponsored by DST, Delhi and conducted by CMS for five weeks during the period 29th May to 30th June 2000. In the second School the main lectures were given by Dr. H.L. Manocha of Delhi, Dr. S. Bhargava of Mysore, Dr. K. Srinivasa Rao and Dr. R. Jagannathan of Chennai and Dr. A.M. Mathai of Montreal, Canada. Supplementary lectures were given and problem sessions were supervised by Dr. K.S.S. Nambooripad and Dr. S.R. Chettiyar of CMS, Dr. R.N. Pillai (retired), Dr. T.S.K. Moothathu and Dr. Yageen Thomas of the University of Kerala and Dr. E. Krishnan of the University College, Trivandrum. Lecture Notes were brought out as Publication No. 31 of CMS soon after the School was completed.

The Third S.E.R.C. School on Special Functions and Functions of Matrix Argument: Recent Developments and Recent Applications in Statistics and Astrophysics, sponsored by DST, Delhi, will be conducted for five weeks from 14th March 2005 to 15th April 2005 by CMS at its Pala Campus, Kerala, India. This time DST, Delhi, wanted the Lecture Notes to be collected from the main lecturers in advance, compiled and distributed prior to the beginning of the School. The main lectures in the Third S.E.R.C. School will be given by Professor Dr. Hans Haubold of the Outer Space Division of the United Nations, Dr. Serge B. Provost, Professor of Actuarial Sciences and Statistics of the University of Western Ontario, Canada, Dr. R.K. Saxena of Jodhpur, India, Dr. S. Bhargava of Mysore, India and Dr. A.M.Mathai of Montreal, Canada. Supplementary lectures will be given by Dr. A. Sukumaran Nair (Chairman, CMS), Dr. K.S.S. Nambooripad (Director-in-Charge, Trivandrum Campus), Dr.K.K. Jose (Director-in-Charge, Pala Campus), Dr. R.N. Pillai, Dr. Yageen Thomas, Dr. M.R. Kaimal, Dr. M.I. Jinnah, Dr. V.K. Sharma (VSSC), Dr. V. Seethalakshmi, Dr. Alice Thomas, Dr. E. Sandhya, Dr. S. Satheesh and Dr. K. Jayakumar. Problem sessions will be supervised by Dr. Sunil C. Mathew, Dr. Alex Thannippara, Dr. Sebastian George, Dr. Benny Kurian, Dr. T.P. Johnson and Dr. M. S. Samuel, besides the main and supplementary lecturers. Extra training in the use of statistical packages, TEX and Mathematica / Maple will be given by Joy Jacob, Seemon Thomas and K.M. Kurian of the Department of Statistics of St. Thomas College, Pala.

Chapter 1 introduces elementary classical special functions. Gamma, beta, psi, zeta functions, hypergeometric functions and the associated special functions, generalization to Meijer's G and Fox's H-functions are also examined here. Discussion is confined to basic properties and some applications. A brief introduction to statistical distribution theory is also given here. Some recent extensions of Dirichlet integrals and Dirichlet densities are given. A glimpse into multivariable special functions such as Appell's functions and Lauricella functions is also given.

Chapter 2 introduces the problem of density estimation and gives a brief look into some recent results in this area through orthogonal polynomials. Chapter 3 gives a detailed coverage of recent results on Mittag-Leffler functions and fractional calculus.

The first part of Chapter 4 looks into some applications of special functions in various problems in astrophysics. The second part of Chapter 4 is on stochastic processes, especially on Mittag-Leffler processes and related topics of infinite divisibility of random variables, geometric infinite divisibility, α-Laplace distributions and stationary time series. Chapter 5 deals with q-series or hypergeometric series with different bases or what is known in the current literature as basic hypergeometric series. Chapter 5 gives some insights into Ramanujan's work on theta functions, mock theta functions, various types of summation formulae and so on.

Chapter 6 gives an introduction to semigroup and Chapter 7 goes into special functions of matrix argument. Various developments are covered in the discussion. This chapter starts with a discussion of the essentials of the Jacobians of matrix transformations which are needed for the development of functions of matrix argument. Only the real case is covered here.

Since the lecture notes were supplied in advance, only a glimpse of what will be covered in the lectures is given here. More up to date materials are expected to be covered in the various lectures. All the lectures will be followed by serious problem sessions on the materials covered in the respective lectures. These problems are listed as exercises in this publication. There will be written tests as well as personal interviews at the end of each week and the cumulative marks in the written tests and oral examinations will be entered into the final certificate.

Several people have contributed enormously for the success of this S.E.R. C. School and in making this publication possible. Dr. B.D. Acharya, Advisor to Government of India and of the Mathematical Sciences Division of the Department of Science and Technology, New Delhi, is the driving force behind the re-energized mathematical activities in India now. He was kind enough to pursue the matter and get the funds released for running the School as well as for the preparation of this publication. Since the materials for this publication are supplied by the various lecturers, there will be some overlaps. Very obvious inconsistencies are removed but some overlapping materials are left there to make the discussions self-contained. Dr. R.K. Saxena, Dr. Serge B. Provost, Dr. S. Bhargava, Dr. A.M. Mathai and Dr. K. K. Jose are thanked for making their notes available in advance for this publication. Notes from others will be made available to the participants of the School. Since the authors did not follow the guidelines in preparing their notes, uniform style could not be kept in the final printed book. Some inconsistencies were also noted between the citations in the text and the reference list. Since there was no time, these

could not be corrected. The material was typeset at CMS office by K.H. Soby, Joy Jacob, Seemon Thomas, Dr. Sebastian George, Dr. K. K.Jose and myself.

It may be mentioned here that, apart from the Publications Series, CMS also brings out books in its Modules Series and Mathematical Sciences for the General Public Series. Three books in the Modules Series are brought out and two books, giving brief biographies of World Mathematicians, are already brought out in the Mathematical Sciences for the General Public Series.

CMS would like to express its sincere thanks to the main lecturers Dr. Hans Haubold, Dr. Serge Provost, Dr. R.K. Saxena, Dr. S. Bhargava and Dr. A.M. Mathai, and the supplementary lecturers Dr. A. Sukumaran Nair, Dr. K.S.S. Nambooripad, Dr. K.K. Jose, Dr. M.R. Kaimal. Dr. M.I. Jinnah, Dr. Yageen Thomas, Dr. V.K. Sharma, Dr. K. Jayakumar, Dr. V. Seethalakshmi, Dr. Alice Thomas, Dr. E. Sandhya and Dr. V. Satheesh, and the problem session leaders Dr. Alex Thannippara, Dr. Sebastian George, Dr. Benny Kurian, Dr. T.P. Johnson, Dr.M.S. Samuel and Dr. Sunil C. Mathew. CMS would like to thank Barbara Haubold and Alexander Haubold for sparing their time for associated lectures for the general public and Joy Jacob, Seemon Thomas and K.M. Kurian for running special training programs on the use of statistical packages, TEX and Mathematica/Maple.

We express our sincere gratitude to the Management and principal, St. Thomas College, Pala for providing all facilities in the college for the successful conduct of the School. The enthusiasm and co-operation shown by Rev.Dr. M.M. Mathew, Principal and Rev.Dr. Mathew John K, Bursar, St. Thomas College, Pala have contributed significantly for the success of the School. The natural beauty of the serene surrounding of the Pastoral Institute and the College campus will be a lasting memory for the participants. CMS would like to thank each and everyone at the Pastoral Institute for making the participants' stay comfortable. The Director and the sisters there deserve special thanks. CMS is grateful to the Manager, Queens Restaurant, Kanjirapplly who was kind enough to make available all the facilities at the restaurant for the participants pf the School as well as for the guest faculty. St. Thomas College canteen authorities have supplied coffee/tea for the participants. The faculty and students from the Departments of Mathematics, Statistics and Physics at St. Thomas College Pala did a lot of voluntary work. CMS would like to thank each and every one who directly or indirectly helped to run the School and make it a success.

Pala, Kerala, India — Dr.A.M. Mathai
15th February 2005 — Director of the School and Director of CMS

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GLOSSARY OF SYMBOLS

$(a)_n$	Pochammer symbol	Notation 1.0.1	1
$n!$	factorial n	Notation 1.0.2	2
$\binom{n}{r}$	combinations	Notation 1.0.3	2
$\Gamma(z)$	gamma z	Notation 1.1.1	3
γ	Euler's constant	Notation 1.1.2	3
$\Re(\cdot)$	real part of $(\cdot)$	Definition 1.1.5	4
$\psi(z)$	psi function	Notation 1.2.1	7
$\zeta(\rho, a)$	generalized zeta function	Notation 1.2.2	7
$\zeta(\rho)$	Riemann zeta function	Notation 1.2.2	7
$x\|y$	x given y	Section 1.3.2	13
$E(x^h)$	expected value	Example 1.4.1	18
$B(\alpha, \beta)$	beta function	Notation 1.6.1	18
$\mathrm{Var}(x)$	variance of x	Exercise 1.4.3	23
$\mathrm{Cov}(x, y)$	covariance between x and y	Exercise 1.4.5	24
${}_pF_q(\cdot)$	hypergeometric function	Notation 1.5.1	24
$\gamma(a, x), \Gamma(a, x)$	incomplete gamma	Example 1.5.1	25
$b(\alpha, \beta; t), I_x(\alpha, \beta)$	incomplete beta	Example 1.5.2	26
$G^{m,n}_{p,q}(\cdot)$	Meijer's G-function	Notation 1.6.1	31
$J_\nu(z)$	Bessel functions	Exercises 1.6.4	36
$I_\nu(z)$	Bessel functions	Exercises 1.6.5	36
$M_{\mu,\nu}(z), W_{\mu,\nu}(z)$	Whittaker function	Exercise 1.6.6	37
$H^{m,n}_{p,q}(z)$	H-function	Notation 1.7.1	37
$E_\alpha(z)$	Mittag-Leffler function	Exercise 1.7.3	40
$E_{\alpha,\beta}(z)$	Mittag-Leffler function, generalized	Exercise 1.7.5	40
$D(\alpha_1, ..., \alpha_k; \alpha_{k+1})$	Dirichlet function	Notation 1.8.1	41
f_A, f_B, f_C, f_D	Lauricella functions	Section 1.9	47
F_1, F_2, F_3, F_4	Appell's functions	Section 1.9	48
$P_k[x]$	Legendre polynomial	Equation (2.2.1)	58
$\mu_y[k]$	k-th moment of y	Equation (2.2.6)	59

$L_j[v,x]$	Laguerre polynomial	Equation (2.3.5)	61
$T_i[x]$	orthogonal polynomial	Equation (2.5.1)	65
$JacobiP[.,.,.]$	Jacobi polynomial	Equation (2.5.9)	67
$HermiteH[.,.]$	Hermite polynomial	Equation (2.5.14)	67
$h_r(z,n)$	hyperbolic function	Definition 3.1.3	72
$k_r(z,n)$	trigonometric function	Definition 3.1.4	73
$erfc(z)$	error function	Definition 3.1.5	73
$erf(z)$	error function	Definition 3.1.5	73
$E_t(\nu,a)$	Mellin-Ross function	Definition 3.1.6	73
$F(\beta,t)$	Robotov's function	Definition 3.1.7	73
${}_p\Psi_q(\cdot)$	Wright function	Section 3.2.2	77
$E^{\delta}_{\beta,\gamma}(z)$	generalized Mittag-Leffler function	Section 3.3	85
${}_aI^n_x, {}_aD^{-n}_x$	fractional integrals	Notation 3.4.1	92
I^{α}_{a+}	Riemann fractional integral	Notation 3.4.2	94
${}_xI^{\alpha}_b, {}_xD^{-\alpha}_b, I^{\alpha}_{b-}$	Riemann fractional integral	Notation 3.4.3	94
${}_xW^{\alpha}_{\infty}, {}_xI^{\alpha}_{\infty}$	Weyl fractional integral	Notation 3.4.5	98
${}_xD^{\alpha}_{\infty}, D^{\alpha}_{\infty}$	Weyl fractional integral	Notation 3.4.6	98
${}_{\infty}W^{\alpha}_x, I^{\alpha}_x$	Weyl integrals	Notation 3.4.7	100
$\{\alpha\}$	fractional part of α	Notation 3.5.1	111
$[\alpha]$	integer part of α	Notation 3.5.2	111
$D^{\alpha}_{x+}, {}_aD^{\alpha}_x$	fractional derivatives	Notation 3.5.3	111
$D^{\alpha}_{b-}, {}_bD^{\alpha}_x$	fractional derivatives	Notation 3.5.4	111
${}_aI^{-\alpha}_x, ({}_aI_x)^{-1}$	fractional derivatives	Definition 3.5.2	112
${}_bI^{-\alpha}_x, ({}_bI_x{}^{\alpha})^{-1}$	fractional derivatives	Definition 3.5.2	112
$R_{\odot}$	solar radius	Section 4.1	122
$R_{\odot}$	solar mass	Section 4.1	122
$F * F$	convolution	Definition 4.3.1	128
$g.i.d.$	geometrically infinitely divisible	Section 4.3.2	129
$c.m.d.$	complete monotone derivative	Section 4.3.3	130
$p.g.f.$	probability generating function	Section 4.3.4	132
$g.s.s.$	geometrically strictly stable	Section 4.3.6	132
MA	moving average	Section 4.4.1	138

AR	autoregressive	Section 4.4.1	138
$i.i.d.$	independently and identically distributed	Section 4.4.2	138
$TMLAR$	tractable Mittag-Leffler AR	Section 4.5.1	144
$(.,.)_\infty$	product representation	Section 5.1.2	162
$X > O$	definiteness of a matrix	Section 7.0	227
I	identity matrix	Section 7.0	227
$\mathrm{tr}(\cdot)$	trace of $(\cdot)$	Section 7.0	227
$(\|(\cdot)\|$	determinant of $(\cdot)$	Section 7.0	227
$\mathrm{d}X$	wedge product of differentials in X	Section 7.1	227
$\frac{\partial}{\partial X}$	matrix differential operator	Section 7.1	228
$J(Y : X) = J$	Jacobian	Section 7.1	228
$\mathrm{d}x \wedge \mathrm{d}y$	skew symmetric product	Section 7.1	229
$\Gamma_p(\alpha)$	matrix-variate real gamma	Section 7.2	235
$B_p(\alpha, \beta)$	matrix-variate real beta	Definition 7.2.2	236
$L_f(\cdot)$	Laplace transform of f	Section 7.3	241
$C_K(Z)$	zonal polynomial	Section 7.4.2	248
$(a)_K$	generalized Pochammer symbol	Section 7.4.2	248